

A GEOMETRIC APPROACH TO COMPLETE REDUCIBILITY

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ABSTRACT. Let G be a connected reductive linear algebraic group. We use geometric methods to investigate G -completely reducible subgroups of G , giving new criteria for G -complete reducibility. We show that a subgroup of G is G -completely reducible if and only if it is strongly reductive in G ; this allows us to use ideas of R.W. Richardson and Hilbert–Mumford–Kempf from geometric invariant theory. We deduce that a normal subgroup of a G -completely reducible subgroup of G is again G -completely reducible, thereby providing an affirmative answer to a question posed by J.-P. Serre, and conversely we prove that the normalizer of a G -completely reducible subgroup of G is again G -completely reducible. Some rationality questions and applications to the spherical building of G are considered. Many of our results extend to the case of non-connected G .

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1. INTRODUCTION

Let G be a connected reductive linear algebraic group defined over an algebraically closed field k . Following Serre [38], we say that a (closed) subgroup H of G is *G-completely reducible* (G -cr) provided that whenever H is contained in a parabolic subgroup P of G , it is contained in a Levi subgroup of P ; for an overview of this concept see for instance [37] and [38]. In the case $G = \mathrm{GL}(V)$ (V a finite-dimensional k -vector space) a subgroup H is G -cr exactly when V is a semisimple H -module, so this faithfully generalizes the notion of complete reducibility from representation theory. The concept of G -complete reducibility is part of the philosophy developed by J.-P. Serre, J. Tits and others to extend standard results from the representation theory of algebraic groups by replacing representations $H \rightarrow \mathrm{GL}(V)$ with homomorphisms $H \rightarrow G$, where the target group is an arbitrary reductive algebraic group; see for instance [17], [18], [34], [36], [37], [38], and [40].

In this paper we apply geometric techniques to study G -complete reducibility. We are motivated by the philosophy of R.W. Richardson [33]. His insight was that one can study subgroups of G indirectly by looking at the action of G on G^n by simultaneous conjugation, where $n \geq 1$; in this setting, one can apply ideas from geometric invariant theory, such as the Hilbert–Mumford Theorem. To this end, he introduced the notion of a *strongly reductive* subgroup of G . A closed subgroup H of G is said to be *strongly reductive in G* provided H is not contained in any proper parabolic subgroup of $C_G(S)$, the centralizer of S in G , where S is a maximal torus of $C_G(H)$, [33, Def. 16.1] (this does not depend on the choice of S). He proved that if the subgroup H is topologically generated by h_1, \dots, h_n , then H is strongly reductive in G if and only if the G -orbit of the n -tuple (h_1, \dots, h_n) is closed in G^n , [33, Thm. 16.4]. In general, not every G -completely reducible subgroup is topologically finitely generated, but standard arguments (see Remark 2.9 and Lemma 2.10) show that one can reduce to this case.

Richardson showed that a closed subgroup H of $\mathrm{GL}(V)$ is strongly reductive if and only if V is a semisimple H -module, [33, Lem. 16.2]: thus strong reductivity and complete reducibility are equivalent for subgroups of $\mathrm{GL}(V)$. Our main result, Theorem 3.1, asserts that the latter assertion holds when $\mathrm{GL}(V)$ is replaced by an arbitrary reductive group G . This allows us to apply results on strong reductivity due to Richardson [33] and the second author [21], [22] to study G -complete reducibility. For example, we deduce immediately that a normal subgroup of a G -completely reducible subgroup of G is also G -completely reducible (Theorem 3.10), thereby answering a question of Serre (see [37, p. 24]). Conversely, we prove that if a subgroup H is G -completely reducible, then so is its normalizer $N_G(H)$ (Corollary 3.16).

We then continue our investigations into G -complete reducibility, centering on the following general question. Let $f : H \rightarrow G$ be a homomorphism of reductive groups and let K be a closed subgroup of H . What hypotheses on H , G , and f guarantee that if K is H -completely reducible, then $f(K)$ is G -completely reducible, or vice versa? Using Lemma 2.12(ii), one can often reduce to the case that H is a closed subgroup of G . Given subgroups $K \subseteq H$ of G with H reductive, several of our results in Section 3 provide criteria that ensure that K is

G -cr if and only if it is H -cr (Corollaries 3.21 and 3.22 and Theorem 3.26). We also give further conditions to ensure that K is H -cr, provided it is G -cr (Proposition 3.19), in particular in case (G, H) is a reductive pair in the sense of Definition 3.32 (Theorem 3.35 and Corollary 3.36). When $G = \mathrm{GL}(V)$, the idea is to reduce the problem of determining H -completely reducible subgroups of H to the representation-theoretic problem of determining subgroups of H that act completely reducibly on V . A theorem of Serre (see Theorem 3.41) gives a lower bound on the characteristic $\mathrm{char} k$ to ensure that H -complete reducibility is equivalent to G -complete reducibility for closed subgroups K of H . We give some related results (see Theorem 3.35, Corollary 3.36, Theorem 3.46, and the final paragraph of Section 3), which improve on his bound in some circumstances. The special case of the adjoint representation is discussed in greater detail: see Example 3.37, Corollary 3.42, Remarks 3.43(ii)–(iii), and Theorem 3.46. A number of our results give new criteria for subgroups to be G -completely reducible, such as Theorem 3.10, Theorem 3.14, and Proposition 3.20.

Much of the previous work on G -complete reducibility relies on a detailed investigation into the properties of each of the classical and exceptional simple algebraic groups (cf. [16], [17], [18], and [23]). We see our methods as complementing this approach: generally the geometric arguments are short and uniform, without requiring a case-by-case analysis.

Most of our results can be extended to the case of a non-connected group G with G^0 reductive; this requires the formalism of *R-parabolic subgroups* and *R-Levi subgroups* of G , to be introduced in Section 6. Often important groups associated to a connected group G , such as $\mathrm{Aut} G$ or centralizers or normalizers of subgroups of G , are not connected. For example, to prove Theorem 1.1 of [21] (see Corollary 3.8 below), even just for connected groups, one needs to consider non-connected groups. Likewise, the formalism of R-parabolic subgroups for non-connected groups allows us to deduce Theorem 3.14 immediately from Propositions 3.19 and 3.12 (see Subsection 6.3). Nevertheless, to avoid technicalities we have formulated our results first for connected G : for example, the proof of our main result Theorem 3.1 uses only the standard theory of connected reductive groups and their parabolic subgroups. The extensions to non-connected G , together with the necessary preliminary results on R-parabolic subgroups, are postponed to Section 6. Some of the results in the earlier sections, such as Proposition 3.19, are proved using the cocharacter language of Lemma 2.4 to facilitate their generalizations in Section 6.

The paper is organized as follows. In Section 2 we first recall some standard tools from geometric invariant theory we require for the sequel. We give a characterization of parabolic subgroups and Levi subgroups of G using cocharacters of G . This is followed by some basic results on G -complete reducibility and related notions. We continue by recalling a number of relevant results on strongly reductive subgroups. This subsection is mostly expository, but we believe it is worth reviewing the basics of Richardson's theory, because it is not well known. The background material is not required for the proof of Theorem 3.1, with which we start Section 3, but it is needed for the other results that follow.

The heart of the paper is Section 3 which contains most of our results on G -cr subgroups, as outlined above.

In Section 4 we discuss the building-theoretic approach to G -complete reducibility due to J.-P. Serre, who has shown that if H is a G -cr subgroup of G , then the geometric realization of the fixed point subcomplex of the action of H on the Tits building of G is homotopy

equivalent to a bouquet of spheres. We investigate instances when this subcomplex is itself a building.

Section 5 is concerned with rationality questions for G -cr subgroups. In particular, we show that if G is defined over a perfect field k , then a k -subgroup of G is “ G -completely reducible over k ” in an appropriate sense if and only if it is G -completely reducible.

Finally, in Section 6 we extend some of our results from Sections 2–5 to the case when G is no longer required to be connected.

Our main references for reductive algebraic groups and their parabolic subgroups are [2], [4], and [41].

2. PRELIMINARIES

2.1. Notation. We maintain the notation from the introduction. In particular, G is a reductive algebraic group defined over a field k . Throughout we assume that reductive groups are connected (in Section 6 we use the term non-connected reductive group to mean a possibly non-connected group with reductive identity component). With the exception of Section 5, we assume that k is algebraically closed. We denote the Lie algebra of G by $\text{Lie } G$ or by \mathfrak{g} ; likewise for closed subgroups of G . For a closed subgroup H of G , we denote its identity component by H^0 . The centralizer and normalizer of H in G are $C_G(H) = \{g \in G \mid ghg^{-1} = h \text{ for all } h \in H\}$ and $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$, respectively. If S is a group acting on G by automorphisms, then $C_G(S)$ is the subgroup of S -fixed points of G . Also the centralizer of H in \mathfrak{g} is defined by $\mathfrak{c}_{\mathfrak{g}}(H) = \{x \in \mathfrak{g} \mid \text{Ad } h(x) = x \text{ for all } h \in H\}$. By a Levi subgroup of G we mean a Levi subgroup of a parabolic subgroup of G . The unipotent radical of a closed subgroup H of G is denoted by $R_u(H)$. Let S be a torus of G . Then $C_G(S)$ is a Levi subgroup of G , [2, Thm. 20.4]. Conversely, every Levi subgroup of G is of this form, e.g., see Lemma 2.4(ii). We write $Z(G)$ for the center of G .

Let H be a closed (not necessarily connected) subgroup of G normalized by some maximal torus T of G : that is, a *regular* subgroup of G (reductive regular subgroups are often also referred to as *subsystem subgroups*, e.g., see [16], [17]). Let $\Psi = \Psi(G, T)$ denote the set of roots of G with respect to T . In this case the root spaces of \mathfrak{h} relative to T are also root spaces of \mathfrak{g} relative to T , and the set of roots of H with respect to T , $\Psi(H) = \Psi(H, T) = \{\alpha \in \Psi \mid \mathfrak{g}_\alpha \subseteq \mathfrak{h}\}$, is a subset of Ψ , where \mathfrak{g}_α denotes the root space in \mathfrak{g} corresponding to α . If $\text{char } k$ does not divide any of the structure constants of the Chevalley commutator relations of G , then $\Psi(H)$ is closed under addition in Ψ in the sense that if $m, n \in \mathbb{N}$ and $\alpha, \beta \in \Psi(H)$ with $m\alpha + n\beta \in \Psi$, then $m\alpha + n\beta \in \Psi(H)$. If H is reductive and regular, then $\Psi(H)$ is a semisimple subsystem of Ψ .

Fix a Borel subgroup B of G containing T and let $\Sigma = \Sigma(G, T)$ be the set of simple roots of Ψ defined by B . Then $\Psi^+ = \Psi(B)$ is the set of positive roots of G . For $\beta \in \Psi^+$ write $\beta = \sum_{\alpha \in \Sigma} c_{\alpha\beta}\alpha$ with $c_{\alpha\beta} \in \mathbb{N}_0$. A prime p is said to be *good* for G if it does not divide $c_{\alpha\beta}$ for any α and β , [42].

Recall that a linear algebraic group S , not necessarily connected, is said to be *linearly reductive* if every rational representation of S is semisimple. It is well known that in characteristic zero, S is linearly reductive if and only if S^0 is reductive. In characteristic $p > 0$, S is linearly reductive if and only if every element of S is semisimple if and only if S^0 is a torus and $|S/S^0|$ is coprime to p , see [28, §4, Thm. 2].

2.2. Characteristic zero. The notions of strong reductivity in G and G -complete reducibility are uninteresting in characteristic zero, as a closed subgroup H is strongly reductive in G if and only if H^0 is reductive if and only if H is G -cr (cf. Lemma 2.6). We therefore assume for the remainder of the paper that k has characteristic $p > 0$. Nevertheless, all of our results hold in characteristic zero with the obvious modifications.

2.3. Geometric invariant theory. We recall some results from geometric invariant theory required in the sequel, see [29, Ch. 3], [1, §2]. Let G be a reductive group acting on an affine variety X (we assume all actions are left actions).

Definition 2.1. For $x \in X$ let $G \cdot x$ denote the G -orbit of x in X and $C_G(x)$ the stabilizer of x in G . Let $Z = \bigcap_{x \in X} C_G(x)$ be the kernel of the action of G on X . Following [33, 1.4] we say that $x \in X$ is a *stable point* for the action of G or a *G -stable point* provided the orbit $G \cdot x$ is closed in X and $C_G(x)/Z$ is finite.

Definition 2.2. Let $\phi : k^* \rightarrow X$ be a morphism of algebraic varieties. We say that $\lim_{t \rightarrow 0} \phi(t)$ exists if there exists a morphism $\widehat{\phi} : k \rightarrow X$ (necessarily unique) whose restriction to k^* is ϕ ; if this limit exists, then we set $\lim_{t \rightarrow 0} \phi(t) = \widehat{\phi}(0)$.

By $Y(G)$ we denote the set of all cocharacters $\lambda : k^* \rightarrow G$ of G . If $\lambda \in Y(G)$ and $g \in G$, then we define $g \cdot \lambda \in Y(G)$ by $(g \cdot \lambda)(t) = g\lambda(t)g^{-1}$; this gives a left action of G on $Y(G)$. It follows easily from Definition 2.2 that if $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists for a cocharacter $\lambda \in Y(G)$, then this limit belongs to the closure $\overline{G \cdot x}$ of $G \cdot x$ in X . The following result, known as the Hilbert–Mumford Theorem [14, Thm. 1.4], gives a converse to this.

Theorem 2.3. *Let G be a reductive group acting on an affine variety X , and let $x \in X$. For any y in the closure of $G \cdot x$, there exists $\lambda \in Y(G)$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists and belongs to $G \cdot y$.*

An important tool in the geometric approach to G -complete reducibility is a strengthened version of the Hilbert–Mumford Theorem due to Kempf [14, Thm. 3.4]: roughly, this says that if y belongs to the complement $\overline{G \cdot x} \setminus G \cdot x$, then there is a *canonical* way of choosing a cocharacter λ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ lies in $G \cdot y$, a so-called “optimal” λ . We refer to this theorem and its corollaries as the Hilbert–Mumford–Kempf Theorem. It is the main ingredient in the proof of Proposition 2.14 and underlies the rationality result [14, Thm. 4.2], which is used indirectly in our proof of Theorem 5.8.

We require the characterization of parabolic subgroups of G in terms of cocharacters of G , see [33, 2.1–2.3] and [41, Prop. 8.4.5]:

Lemma 2.4. *Given a parabolic subgroup P of G and any Levi subgroup L of P , there exists $\lambda \in Y(G)$ such that the following hold:*

- (i) $P = P_\lambda := \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$.
- (ii) $L = L_\lambda := C_G(\lambda(k^*))$.
- (iii) The map $c_\lambda : P_\lambda \rightarrow L_\lambda$ defined by

$$c_\lambda(g) := \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1}$$

is a surjective homomorphism of algebraic groups. Moreover, L_λ is the set of fixed points of c_λ and $R_u(P_\lambda)$ is the kernel of c_λ .

Conversely, given any $\lambda \in Y(G)$ the subset P_λ defined as in part (i) is a parabolic subgroup of G , L_λ is a Levi subgroup of P_λ , and the map c_λ as defined in part (iii) has the described properties. Moreover, P_λ is a proper subgroup if and only if $\lambda(k^*) \not\subseteq Z(G)$.

Let H be a reductive subgroup of G . There is a natural inclusion $Y(H) \subseteq Y(G)$ of cocharacter groups. If necessary, we distinguish between the ambient groups we are working in by writing $P_\lambda(H)$, $P_\lambda(G)$, etc., for $\lambda \in Y(H)$; usually we just write P_λ rather than $P_\lambda(G)$. It is obvious from Lemma 2.4 that if $\lambda \in Y(H)$, then $P_\lambda(H) = P_\lambda(G) \cap H$ and similarly for $L_\lambda(H)$ and $R_u(P_\lambda(H))$. We record this in our next result.

Corollary 2.5. *Let H be a reductive subgroup of G . If Q is a parabolic subgroup of H and M is a Levi subgroup of Q , then there exists a parabolic subgroup P of G and a Levi subgroup L of P such that $Q = P \cap H$, $M = L \cap H$ and $R_u(Q) = R_u(P) \cap H$.*

We say that a closed subgroup H of G is *topologically generated* by $h_1, \dots, h_n \in G$ provided that H is the Zariski closure of the subgroup of G generated by these elements. As we are concerned with topologically finitely generated subgroups of G , we are interested in the diagonal action of G on the affine variety G^n (for some $n \in \mathbb{N}$) by simultaneous conjugation:

$$g \cdot (g_1, \dots, g_n) := (gg_1g^{-1}, \dots, gg_ng^{-1}).$$

Let $(h_1, \dots, h_n) \in G^n$ and let H be the subgroup of G topologically generated by h_1, \dots, h_n . Let $\lambda \in Y(G)$. It follows easily from Definition 2.2 that $\lim_{t \rightarrow 0} \lambda(t) \cdot (h_1, \dots, h_n)$ exists if and only if $\lim_{t \rightarrow 0} \lambda(t)h_i\lambda(t)^{-1}$ exists for $1 \leq i \leq n$, and this is the case if and only if all of the h_i lie in P_λ , cf. Lemma 2.4(i). Since the h_i topologically generate H , this occurs if and only if H is a subgroup of P_λ , and in this case, $c_\lambda(h_1), \dots, c_\lambda(h_n)$ topologically generate $c_\lambda(H)$, where c_λ is the map defined in Lemma 2.4(iii).

2.4. Basic properties of G -cr, G -ir, and G -ind subgroups. The concept of G -complete reducibility is a relative notion depending on the embedding of the subgroup into G . Note that G is trivially a G -cr subgroup of itself. If H is a closed G -cr subgroup of G , then H^0 is reductive, [37, Property 4]. In characteristic zero the converse holds. This follows from a well-known result due to G. Mostow [24]. More generally, we have the following result (e.g., see [13, Lem. 11.24]):

Lemma 2.6. *Let S be a linearly reductive subgroup of G . Then S is G -completely reducible.*

An important class of G -cr subgroups consists of those that are not contained in any proper parabolic subgroup of G at all (they are trivially G -cr). Following Serre, we call them *G -irreducible* (G -ir), [38]. As with the concept of G -complete reducibility, this terminology stems from the fact that it coincides with the usual notion of irreducibility in the classical case $G = \mathrm{GL}(V)$. Observe that every overgroup of a G -irreducible subgroup of G is itself G -irreducible. Moreover, following Serre [38], we say that a subgroup H of G is *G -indecomposable* (G -ind) provided that H is not contained in any Levi subgroup of any proper parabolic subgroup of G . Again, in the classical case $G = \mathrm{GL}(V)$ this coincides with the usual property of V being an indecomposable H -module.

Note that a closed subgroup H of G is strongly reductive in G if and only if H is $C_G(S)$ -ir, where S is a maximal torus of $C_G(H)$.

Part (i) of our next result is due to Slodowy in the special case $G = \mathrm{GL}(V)$, [39, Lem. 11].

Corollary 2.7. *Let $K \subseteq H$ be closed subgroups of G with H reductive.*

- (i) *If K is G -irreducible, then it is H -irreducible.*
- (ii) *If K is G -indecomposable, then it is H -indecomposable.*

Proof. It follows directly from Corollary 2.5 that if K is contained in a proper parabolic subgroup of H , then it is contained in a proper parabolic subgroup of G as well, giving (i). Similarly, if K is contained in a proper Levi subgroup of H , then it is also contained in a proper Levi subgroup of G , again by Corollary 2.5, giving (ii). \square

Remarks 2.8. (i). If H is not G -irreducible, then taking $K = H$ gives a trivial counterexample to the converse of Corollary 2.7(i). Example 3.44 below (taking $K = \mathrm{PGL}_p(k)$, $H = \mathrm{PGL}_m(k)$, and $G = \mathrm{GL}(\mathfrak{h})$) shows that the converse of Corollary 2.7(i) can fail even when H is G -ir. Likewise, if H is not G -ind, taking $K = H$ again gives a trivial counterexample to the converse of Corollary 2.7(ii).

(ii). Serre's Theorem 3.41(ii) below shows that if $\mathrm{char} k$ is sufficiently large and $G = \mathrm{GL}(V)$, then the implication of Corollary 2.7(i) does hold for G -cr and H -cr in place of G -ir and H -ir, respectively. However, this statement does not hold in general, e.g., see Example 3.45 below. Several of the results in Section 3 provide criteria to ensure the analogous implication holds for G -complete reducibility and H -complete reducibility, for instance see Proposition 3.19, Corollary 3.21, Theorem 3.26, Theorem 3.35, and Corollary 3.36.

Remark 2.9. When considering questions of G -complete reducibility, etc., it is technically convenient to work with topologically finitely generated subgroups of G . Unfortunately, not every reductive subgroup of G has this property: for example, if k is the algebraic closure of the prime field \mathbb{F}_p , then every finitely generated subgroup of G is finite. The following argument of Richardson [33, Prop. 16.9] allows us to reduce to the case of topologically finitely generated subgroups of G in our study of G -cr subgroups; it is fundamental in employing the methods from geometric invariant theory outlined in Subsection 2.3. By Theorem 5.3 and Remark 5.4 below, one may assume without loss that k is transcendental over \mathbb{F}_p . Then one can show that any reductive subgroup H of G is topologically finitely generated, cf. [21, Lem. 9.2]. We use this idea repeatedly in what follows.

In the proof of Theorem 5.8 we need another method for reducing to the topologically finitely generated case.

Lemma 2.10. *Let H be a closed subgroup of G . There exists a finitely generated subgroup Γ of H such that for any parabolic subgroup P of G and any Levi subgroup L of P , P contains H if and only if P contains Γ , and L contains H if and only if L contains Γ .*

Proof. It is well known that G has only finitely many conjugacy classes of parabolic subgroups, and each parabolic subgroup P has exactly one P -conjugacy class of Levi subgroups, so we can choose a finite set of representatives P_1, \dots, P_m and L_1, \dots, L_n for the set of G -conjugacy classes of parabolic subgroups and Levi subgroups respectively. For any $i = 1, \dots, m$ and $j = 1, \dots, n$ and any $H' \subseteq H$, set $C_i(H') = \{g \in G \mid H' \subseteq gP_i g^{-1}\}$ and $D_j(H') = \{g \in G \mid H' \subseteq gL_j g^{-1}\}$. Each $C_i(H')$ and $D_j(H')$ is closed, and if $H' \subseteq H''$, then $C_i(H'') \subseteq C_i(H')$ and $D_j(H'') \subseteq D_j(H')$. The descending chain condition on closed subsets of G , together with a simple application of Zorn's Lemma, implies that for some finitely generated subgroup $\Gamma \subseteq H$, we have $C_i(\Gamma) = C_i(H)$ and $D_j(\Gamma) = D_j(H)$ for every i and j . The result now follows. \square

We want to investigate how the properties of G -complete reducibility, G -irreducibility, and G -indecomposability behave under homomorphisms of the ambient groups, cf. [38, Cor. 4.3]. The next result and part (ii) of Lemma 2.12 answer this question for epimorphisms.

Lemma 2.11. *Let $f : G_1 \rightarrow G_2$ be an isogeny of reductive groups and let $\lambda \in Y(G_1)$. Set $\mu = f \circ \lambda \in Y(G_2)$. Then*

- (i) $f(P_\lambda) = P_\mu$, $f(L_\lambda) = L_\mu$;
- (ii) $f^{-1}(P_\mu) = P_\lambda$, $f^{-1}(L_\mu) = L_\lambda$;
- (iii) L_λ is a proper subgroup of G_1 if and only if L_μ is a proper subgroup of G_2 ;
- (iv) $f(R_u(P_\lambda)) = R_u(P_\mu)$.

Proof. Suppose that we have a connected subset S of G_1 and an element $x \in G_1$ such that $f(x)$ centralizes $f(S)$. Then $[x, S] \subseteq \ker f$, and we deduce that $[x, S] = \{1\}$, as $\ker f$ is finite. This implies that $f(L_\lambda) = f(C_{G_1}(\lambda(k^*))) = C_{G_2}(f(\lambda(k^*))) = C_{G_2}(\mu(k^*)) = L_\mu$, and part (iii) also follows.

If B is a Borel subgroup of G_1 , then $f(B)$ is a Borel subgroup of G_2 ([2, Prop. 16.7]). This implies that $f(P_\lambda)$ is a parabolic subgroup of G_2 . It is clear from the definition of a limit, Definition 2.2, that $f(P_\lambda) \subseteq P_\mu$. As $f(P_\lambda)$ contains $f(L_\lambda) = L_\mu$, a Levi subgroup of P_μ , [4, Prop. 4.4(c)] implies that $f(P_\lambda) = P_\mu$, which completes the proof of part (i).

Next we observe that $\ker f$ is central in G_1 , so it is contained in any Levi subgroup of G_1 . Thus (ii) follows easily from (i).

It is clear that $f(R_u(P_\lambda))$ is a closed connected unipotent normal subgroup of P_μ , so $f(R_u(P_\lambda)) \subseteq R_u(P_\mu)$. Since L_λ and L_μ are Levi subgroups of P_λ and P_μ , respectively, part (i) together with a simple dimension-counting argument implies that $f(R_u(P_\lambda)) = R_u(P_\mu)$, as required. \square

Note that if $f : G_1 \rightarrow G_2$ is an isogeny of reductive groups, then any $\mu \in Y(G_2)$ is of the form $f \circ \lambda$ for some $\lambda \in Y(G_1)$.

Let $f : G_1 \rightarrow G_2$ be a homomorphism of algebraic groups. We say that f is *non-degenerate* provided $(\ker f)^0$ is a torus, cf. [38, Cor. 4.3]. In particular, f is non-degenerate if f is an isogeny.

Lemma 2.12. *Let G_1 and G_2 be reductive groups.*

- (i) *Let H be a closed subgroup of $G_1 \times G_2$. Let $\pi_i : G_1 \times G_2 \rightarrow G_i$ be the canonical projection for $i = 1, 2$. Then H is $(G_1 \times G_2)$ -cr ($(G_1 \times G_2)$ -ir, $(G_1 \times G_2)$ -ind) if and only if $\pi_i(H)$ is G_i -cr (G_i -ir, G_i -ind) for $i = 1, 2$.*
- (ii) *Let $f : G_1 \rightarrow G_2$ be an epimorphism. Let H_1 and H_2 be closed subgroups of G_1 and G_2 , respectively.*
 - (a) *If H_1 is G_1 -cr (G_1 -ir, G_1 -ind), then $f(H_1)$ is G_2 -cr (G_2 -ir, G_2 -ind).*
 - (b) *If f is non-degenerate, then H_1 is G_1 -cr (G_1 -ir, G_1 -ind) if and only if $f(H_1)$ is G_2 -cr (G_2 -ir, G_2 -ind), and H_2 is G_2 -cr (G_2 -ir, G_2 -ind) if and only if $f^{-1}(H_2)$ is G_1 -cr (G_1 -ir, G_1 -ind).*

Proof. (i). The parabolic subgroups of $G_1 \times G_2$ are precisely the subgroups of the form $P_1 \times P_2$, where P_i is a parabolic subgroup of G_i . If $P_1 \times P_2$ is such a subgroup, then the Levi subgroups of $P_1 \times P_2$ are precisely the subgroups of the form $L_1 \times L_2$, where L_i is a Levi subgroup of P_i . Part (i) now follows.

(ii). Let $N = \ker f$. It is standard that there exists a closed reductive subgroup M_1 of $C_{G_1}(N^0)$ with $M_1 \cap N^0$ finite and $G_1 = M_1 N^0$ (cf. the proof of Lemma 6.14). It follows from Lemma 2.11(i)–(iii) that the required result holds for isogenies, so we can assume that N is connected, $G_1 = M_1 \times N$, and f is the projection from G_1 to M_1 . Parts (a) and (b) now follow from part (i); note that if N is a torus, then any closed subgroup of N is trivially N -cr, N -ir, and N -ind. \square

We require the following characterization of topologically finitely generated G -irreducible subgroups of G in terms of stability due to R.W. Richardson, [33, Prop. 16.7].

Proposition 2.13. *Let $x_1, \dots, x_n \in G$ and let H be the subgroup of G that is topologically generated by x_1, \dots, x_n . Then H is G -irreducible if and only if (x_1, \dots, x_n) is a stable point under the diagonal action of G on G^n , that is if and only if the G -orbit of (x_1, \dots, x_n) in G^n is closed and $C_G(H)^0 = Z(G)^0$.*

Observe that the last equivalence in Proposition 2.13 follows from the definition of stability, Definition 2.1, as $C_G(H)$ is the stabilizer in G of $(x_1, \dots, x_n) \in G^n$.

2.5. Strongly reductive subgroups of G . If H is a strongly reductive subgroup of G , then H^0 is reductive, [33, Lem. 16.3], or [21, §6]. In characteristic zero, the converse also holds, [33, §16], or [21, Prop. 6.6]. However, in positive characteristic, this is a more subtle notion, which depends on the embedding of H into G . Trivially, G is a strongly reductive subgroup of itself. We require several results on strong reductivity, the first of which is due to the second author, [22, Thm. 2].

Proposition 2.14. *Let H be a strongly reductive subgroup of G and let N be a closed normal subgroup of H . Then N is strongly reductive in G .*

The following two results are due to R.W. Richardson [33, Prop. 16.9, Thm. 16.4].

Proposition 2.15. *Let S be a linearly reductive group acting on the reductive group G by automorphisms and let $H = C_G(S)^0$. Suppose K is a closed subgroup of H . Then K is strongly reductive in H if and only if it is strongly reductive in G .*

Recall that $H = C_G(S)^0$ is reductive, [32, Prop. 10.1.5].

The following result gives a geometric interpretation for topologically finitely generated algebraic subgroups of G that are strongly reductive in G .

Proposition 2.16. *Let $x_1, \dots, x_n \in G$ and let H be the subgroup of G that is topologically generated by x_1, \dots, x_n . Then H is strongly reductive in G if and only if the G -orbit of (x_1, \dots, x_n) under the diagonal action of G on G^n is closed.*

Observe that the case $n = 1$ is simply the characterization of semisimple elements in G , [42, Cor. 3.6].

Using the Hilbert–Mumford Theorem 2.3 and the map c_λ from Lemma 2.4(iii), we give another characterization of strong reductivity.

Lemma 2.17. *Let H be a closed subgroup of G . Then H is strongly reductive in G if and only if for every cocharacter λ of G with $H \subseteq P_\lambda$, there exists $g \in G$ such that $c_\lambda(h) = ghg^{-1}$ for every $h \in H$.*

Proof. For simplicity, we assume that H is topologically finitely generated, say by h_1, \dots, h_n . (This suffices for the application to Theorem 3.46 below; the general case follows from a slightly more complicated argument involving Lemma 2.10.) For any cocharacter λ such that $H \subseteq P_\lambda$, the subgroup $c_\lambda(H)$ is topologically generated by the elements of the tuple $(h'_1, \dots, h'_n) := \lim_{t \rightarrow 0} \lambda(t) \cdot (h_1, \dots, h_n)$, and this tuple belongs to the closure of the G -orbit $G \cdot (h_1, \dots, h_n)$ in G^n .

If H is strongly reductive in G , then $G \cdot (h_1, \dots, h_n)$ is closed, by Proposition 2.16, so $(h'_1, \dots, h'_n) = g \cdot (h_1, \dots, h_n)$ for some $g \in G$, so $c_\lambda(h_i) = h'_i = gh_i g^{-1}$ for every i . This implies that $c_\lambda(h) = ghg^{-1}$ for every $h \in H$.

Conversely, suppose that for every $\lambda \in Y(G)$ with $H \subseteq P_\lambda$, there exists $g \in G$ with $c_\lambda(h) = ghg^{-1}$ for every $h \in H$. Then for every such cocharacter λ , there exists $g \in G$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot (h_1, \dots, h_n) = g \cdot (h_1, \dots, h_n)$. The Hilbert–Mumford Theorem 2.3 now implies that $G \cdot (h_1, \dots, h_n)$ is closed, so H is strongly reductive in G , again by Proposition 2.16. \square

Remark 2.18. The conclusion of Lemma 2.17, that H and $c_\lambda(H) \subseteq L_\lambda$ are G -conjugate, does not imply *a priori* that H is itself in a Levi subgroup of P_λ . That this is indeed the case is the content of the reverse implication of Theorem 3.1 below.

2.6. Non-closed subgroups. Observe that the notions of strong reductivity in G , G -complete reducibility, etc., apply also to non-closed subgroups. Clearly, a subgroup H of G is G -cr if and only if its Zariski closure \overline{H} is, and similarly for G -irreducibility, G -indecomposability and strong reductivity in G . For convenience we only consider closed subgroups of G in the sequel; note that all the results we give hold for non-closed subgroups as well, wherever this makes sense.

3. G -COMPLETE REDUCIBILITY

3.1. G -complete reducibility and strong reductivity in G . The basis for our study of G -completely reducible groups is

Theorem 3.1. *Let G be reductive and suppose H is a closed subgroup of G . Then H is G -completely reducible if and only if H is strongly reductive in G .*

Proof. Suppose that H is G -cr, and let S be a maximal torus of $C_G(H)$. Now suppose that H is contained in some proper parabolic subgroup Q of $C_G(S)$. There exists a parabolic subgroup P of G such that $Q = C_G(S) \cap P$ ([4, Prop. 4.4(c)]). Note that $S \subseteq P$, because S is central in $C_G(S)$. Since H is G -cr, there is a Levi subgroup L of P such that $H \subseteq L$, and $T := Z(L)^0$ is a torus of $C_P(H)$. Now S is a maximal torus of $C_P(H)$, so $gTg^{-1} \subseteq S$ for some $g \in C_P(H)$. It follows that $C_G(S) \subseteq C_G(gTg^{-1}) = gC_G(T)g^{-1} = gLg^{-1} \subseteq P$, whence $Q = C_G(S)$, a contradiction. Thus H is strongly reductive in G .

If H is strongly reductive in G and S is a maximal torus of $C_G(H)$, then H is not in any proper parabolic subgroup of $C_G(S)$. Since S is a torus, $L := C_G(S)$ is a Levi subgroup of G . Let Q be a parabolic subgroup of G containing L as a Levi subgroup. Then, since H is in no proper parabolic subgroup of L , it follows from [4, Prop. 4.4(c)] that Q is minimal among all parabolic subgroups containing H . Now let P be a parabolic subgroup of G containing H . Then $H \subseteq P \cap Q$. If P' is a parabolic subgroup of G with $P' \subseteq P$ and M' is a Levi subgroup of P' , then there exists a Levi subgroup M of P such that $M' \subseteq M$. Therefore, we may assume that P is minimal subject to $P \supseteq H$. Since $(P \cap Q)R_u(Q)$ is a parabolic subgroup of

G ([4, Prop. 4.4(b)]) contained in Q , the minimality of Q implies that $Q = (P \cap Q)R_u(Q)$. By [4, Prop. 4.4(b)] it follows that P contains a Levi subgroup, M_Q say, of Q . By symmetry, Q contains a Levi subgroup, M_P say, of P . It follows that $P \cap Q$ contains a common Levi subgroup of both P and Q . For, fix Levi subgroups, L_P and L_Q , of P and Q respectively such that $L_P \cap L_Q$ contains a maximal torus of G . Then we have a decomposition

$$(3.2) \quad P \cap Q = (L_P \cap L_Q)(L_P \cap R_u(Q))(R_u(P) \cap L_Q)(R_u(P) \cap R_u(Q))$$

(this is standard; see Lemma 6.2(iii) below for a proof). Moreover, $R_u(P \cap Q)$ is the product of the last three factors in (3.2). Since M_P is reductive, $M_P \cap R_u(P \cap Q)$ is trivial. Thus there is a bijective homomorphism from M_P onto a subgroup of $L_P \cap L_Q$. Since L_P and M_P are P -conjugate, we get $L_P \subseteq L_Q$. Likewise, $L_Q \subseteq L_P$. It follows that $M := L_P = L_Q$ is a Levi subgroup of both P and Q , as claimed.

Let P^- be the unique parabolic subgroup of G opposite to P , such that $P \cap P^- = M$. We may factor $R_u(Q)$ as follows:

$$(3.3) \quad R_u(Q) = (R_u(Q) \cap R_u(P^-))(R_u(Q) \cap R_u(P)).$$

For the set of roots of G with respect to some maximal torus of G in M decomposes as a disjoint union $\Psi(R_u(P)) \cup \Psi(M) \cup \Psi(R_u(P^-))$. The decomposition (3.3) follows, as $R_u(Q) \cap M$ is trivial.

Now, since L and M are Levi subgroups of Q , there exists x in $R_u(Q)$ with $xMx^{-1} = L$. Thanks to (3.3), we may write $x = yz$ with $y \in R_u(Q) \cap R_u(P^-)$ and $z \in R_u(Q) \cap R_u(P)$. Then, since $z \in P \cap Q$, we have $zMz^{-1} \subseteq P \cap Q$. Thus we may assume that $z = 1$. Now, as $y \in P^-$, we see that $L = yMy^{-1}$ lies in P^- . Thus H lies in P^- . Consequently, $H \subseteq P \cap P^- = M$. It follows that H is G -cr, as required. \square

Remark 3.4. Serre has observed that the definitions of strong reductivity in G and G -complete reducibility make sense for an arbitrary, possibly non-smooth, subgroup subscheme of G . The proof of Theorem 3.1 goes through virtually unchanged in this setting; note that if P and Q are parabolic subgroups of G , then the scheme-theoretic intersection of P and Q is smooth, see [10, vol. 3, Lem. 4.1.1].

Corollary 3.5. *Let H be a closed subgroup of G . Then the following are equivalent:*

- (i) H is strongly reductive in G ;
- (ii) H is G -completely reducible;
- (iii) H is $C_G(S)$ -irreducible, where S is a maximal torus of $C_G(H)$;
- (iv) for every parabolic subgroup P of G which is minimal with respect to containing H , the subgroup H is L -irreducible for some Levi subgroup L of P ;
- (v) there exists a parabolic subgroup P of G which is minimal with respect to containing H , such that H is L -irreducible for some Levi subgroup L of P .

Proof. The equivalences between (i), (ii), and (iii) follow from Theorem 3.1 and the definition of strong reductivity. It is clear that (iv) implies (v). Given a maximal torus S of $C_G(H)$, $C_G(S)$ is a Levi subgroup of some parabolic subgroup P of G ; if H is $C_G(S)$ -ir, then P is minimal with respect to containing H , by [4, Prop. 4.4(c)], so (iii) implies (v). If there exists a parabolic subgroup P minimal with respect to containing H and H is an L -ir subgroup of some Levi subgroup L of P , then the proof of Theorem 3.1 implies that H is G -cr, so (v) implies (ii). Finally, if H is G -cr, then H is contained in a Levi subgroup L of P , where P

is any parabolic subgroup minimal with respect to containing H . By [4, Prop. 4.4(c)], H is L -ir, so (ii) implies (iv). \square

Remark 3.6. By Corollary 3.5 the study of G -cr subgroups of G reduces to the study of L -ir subgroups of the Levi subgroups L of G (including the case $L = G$).

Using Theorem 3.1 and the results on strong reductivity from Section 2, we immediately deduce results on G -complete reducibility. Our next result, which follows directly from Proposition 2.16 and Theorem 3.1, allows us to use methods from geometric invariant theory to study G -completely reducible subgroups. It is crucial for a number of results to follow.

Corollary 3.7. *Let $x_1, \dots, x_n \in G$ (for some $n \in \mathbb{N}$) and let H be the subgroup of G topologically generated by x_1, \dots, x_n . Then H is G -completely reducible if and only if the orbit of (x_1, \dots, x_n) under the diagonal action of G on G^n is closed.*

Observe that the notions of G -complete reducibility, etc., all apply to finite subgroups of G . Our next result follows immediately from Theorem 3.1 and [21, Thm. 1.2], which is the corresponding result for strongly reductive subgroups of G .

Corollary 3.8. *There is only a finite number c_N of G -conjugacy classes of G -completely reducible subgroups of fixed order N .*

Remarks 3.9. (i). In [15, Prop. 2.1] it is proved, for G simple and of adjoint type, that there is a uniform bound on c_N as in Corollary 3.8 that depends only on N and the type of G , but not on the field k .

(ii). Observe that in general there exist infinitely many G -conjugacy classes of connected G -cr subgroups of bounded dimension. For example, the subtori T_n of $\mathrm{GL}_2(k)$ defined by $T_n := \left\{ \begin{pmatrix} t^n & 0 \\ 0 & t \end{pmatrix} \mid t \in k^* \right\}$ for n a positive integer are pairwise non-conjugate. For a more subtle example involving semisimple subgroups, see [18, Cor. 4.5].

3.2. Normal subgroups of G -cr subgroups. Our next result, which follows immediately from Proposition 2.14 and Theorem 3.1, gives an affirmative answer to Serre's question, [37, p. 24]. The special case when $G = \mathrm{GL}(V)$ is just a particular instance of Clifford Theory.

Theorem 3.10. *Let H be a closed subgroup of G with closed normal subgroup N . If H is G -completely reducible, then so is N . In particular, if H is G -completely reducible, then so is H^0 .*

Remark 3.11. Serre proves a converse to Theorem 3.10 in [37, Property 5] under the assumption that the index of N in H is prime to $\mathrm{char} k = p$. Examples show that this restriction cannot be removed. For instance, let U be a non-trivial finite unipotent subgroup of G . Then, by a construction due to Borel and Tits [5, §3], there exists a parabolic subgroup P of G such that $U \subseteq R_u(P)$. In particular, U is not G -cr, but clearly $U^0 = \{1\}$ is. In Theorem 3.14 below we give a converse of Theorem 3.10 without characteristic restrictions but with the additional assumption that H contains the connected centralizer of N in G .

3.3. Normalizers and centralizers of G -cr subgroups. If $\mathrm{char} k = 0$ and H is a reductive subgroup of G , then $C_G(H)^0$ and $N_G(H)^0$ are also known to be reductive. In positive characteristic this is still true provided H is linearly reductive, thanks to [32, Prop. 10.1.5].

However, it is clearly false in general. For instance, if V is an indecomposable, non-simple H -module with isomorphic top and socle, then the centralizer of H in $\mathrm{GL}(V)$ is not reductive. Nevertheless, as an application of Corollary 3.7, we have the following result.

Proposition 3.12. *Let H be a G -completely reducible subgroup of G . Then $C_G(H)^0$ is reductive. Moreover, let K be a closed subgroup of G satisfying $H^0 C_G(H)^0 \subseteq K \subseteq N_G(H)$. Then K^0 is reductive. In particular, $N_G(H)^0$ is reductive.*

Proof. By Remark 2.9, we may assume that H is topologically finitely generated, say by $x_1, \dots, x_n \in G$. Observe that the stabilizer of (x_1, \dots, x_n) in G is just $C_G(H)$. As H is G -completely reducible, the G -orbit of (x_1, \dots, x_n) in G^n is closed, by Corollary 3.7, whence $G/C_G(H)$ is affine (e.g., [32, Lem. 10.1.3]). Thanks to [31, Thm. A], it follows that $C_G(H)^0$ is reductive.

Now $H^0 \times C_G(H)^0$ is reductive and thus so is its image $H^0 C_G(H)^0$ in G . Since $N_G(H)$ is a finite extension of $H^0 C_G(H)^0$, see [21, Lem. 6.8], the desired result follows. \square

Remark 3.13. The main result in [16, Thm. 1] asserts that if $p > 7$ and H is any reductive subgroup of the simple exceptional group G , then H is G -cr. In fact, the bounds established in loc. cit. are much more detailed and smaller depending on the types of G and H . Using this result, Liebeck and Seitz proceed in [16, Thm. 2] to show that the connected centralizer $C_G(H)^0$ for each such H is again reductive. Proposition 3.12 gives a short proof of this fact for any reductive G .

Theorem 3.14 can be viewed as a partial converse of Theorem 3.10 complementing Serre's result [37, Property 5], cf. Remark 3.11.

Theorem 3.14. *Let H be a G -completely reducible subgroup of G and suppose K is a closed subgroup of G satisfying $H C_G(H)^0 \subseteq K \subseteq N_G(H)$. Then K is G -completely reducible.*

Proof. Let P be a parabolic subgroup of G containing K . Since H is G -cr, there is a Levi subgroup L of P with $L \supseteq H$. Thanks to Lemma 2.4 there exists a cocharacter λ of G such that $P = P_\lambda$ and $L = L_\lambda$. Since $\lambda(k^*) \subseteq C_G(H)^0 \subseteq K$, we see that K is c_λ -stable, where c_λ is the homomorphism defined in Lemma 2.4(iii). We have

$$(3.15) \quad K = (K \cap L)(K \cap R_u(P)).$$

This follows, as every $x \in K \subseteq P = LR_u(P)$ has a unique factorization $x = x_1 x_2$ with $x_1 \in L$ and $x_2 \in R_u(P)$. Since K is c_λ -stable, we have $c_\lambda(x) = x_1 \in K \cap L$ and this implies (3.15). For any $x \in K \cap R_u(P)$ we have a morphism $\phi_x : k \rightarrow K \cap R_u(P)$ given by $\phi_x(t) = \lambda(t)x\lambda(t)^{-1}$ for $t \in k^*$ and $\phi_x(0) = c_\lambda(x) = 1$. Thus the image of ϕ_x is a connected subvariety of $K \cap R_u(P)$ containing 1 and x . We deduce that $K \cap R_u(P)$ is connected. By assumption $H C_G(H)^0 \subseteq K$ and thus $H^0 C_G(H)^0 \subseteq K$. Thus Proposition 3.12 implies that $K \cap R_u(P)$ is trivial. Consequently, thanks to (3.15), we have $K \subseteq L$, as required. \square

The following are immediate consequences of Theorems 3.10 and 3.14.

Corollary 3.16. *Let H be a closed subgroup of G . Then H is G -completely reducible if and only if $N_G(H)$ is.*

Corollary 3.17. *Let H be a closed subgroup of G . If H is G -completely reducible, then so is $C_G(H)$.*

Proof. Since $C_G(H)$ is normal in $N_G(H)$, the result follows from Theorem 3.10 and Corollary 3.16. \square

We observe that in the classical case $G = \mathrm{GL}(V)$, Corollaries 3.16 and 3.17 are just consequences of Clifford Theory and Wedderburn's Theorem.

In general, the converse of Corollary 3.17 is false: e.g., let H be a Borel subgroup of G . However, we do have the following partial converse. The proof follows immediately from Lemma 2.6 and Lemma 3.38 below.

Corollary 3.18. *Let H be a closed subgroup of G . If $C_G(H)$ is G -irreducible, then H is linearly reductive. In particular, H is G -completely reducible.*

The hypothesis of Corollary 3.18 is very restrictive: since $H \subseteq C_G(C_G(H))$, Remark 2.9 and Proposition 2.13 imply that $H^0 \subseteq Z(G)^0$.

Our next result is similar in nature to Theorem 3.14; it also provides another criterion to ensure that if $K \subseteq H \subseteq G$ with H reductive and K is G -cr, then K is also H -cr.

Proposition 3.19. *Let $K \subseteq H$ be closed subgroups of G with H reductive. Suppose that H contains a maximal torus of $C_G(K)$ and that K is G -completely reducible. Then K is H -completely reducible and H is G -completely reducible.*

Proof. Let S be a maximal torus of $C_G(K)$ contained in H . Then S is also a maximal torus of $C_H(K)$. Since K is G -cr, K is $C_G(S)$ -ir, by Corollary 3.5. Applying Corollary 2.7(i) to $K \subseteq C_H(S) \subseteq C_G(S)$, we see that K is $C_H(S)$ -ir. Thus K is H -cr, again by Corollary 3.5.

Let P be a parabolic subgroup of G containing H . Then P also contains K and S . Since K is G -cr and $S \subseteq P$, it follows that $C_G(S)$ is contained in a Levi subgroup of P (cf. the first paragraph of the proof of Theorem 3.1): say $C_G(S) \subseteq L_\lambda$, where $\lambda \in Y(G)$ and $P = P_\lambda$. We have $\lambda(k^*) \subseteq C_G(C_G(S))^0 = Z(C_G(S))^0$. Clearly, $S \subseteq Z(C_G(S))^0$. Since $K \subseteq C_G(S)$, we have $Z(C_G(S))^0 \subseteq C_G(K)$. Because S is a maximal torus of $C_G(K)$, we get the equality $S = Z(C_G(S))^0$. So we have $\lambda(k^*) \subseteq S$ which is contained in H , by assumption; so λ is a cocharacter of H . But now we see that $P_\lambda(H) = P_\lambda \cap H = H$. It follows from Lemma 2.4 that $\lambda(k^*) \subseteq Z(H)$. So $H \subseteq C_G(\lambda(k^*)) = L_\lambda$, as desired. \square

3.4. G -complete reducibility and regular subgroups. Recall from Subsection 2.1 that H is a regular subgroup of G provided it is normalized by a maximal torus of G .

Proposition 3.20. *Let H be a regular reductive subgroup of G . Then H is G -completely reducible.*

Proof. Let T be a maximal torus of G normalizing H . If $T \subseteq H$, then, since $C_G(T) = T$ and T is G -cr by Lemma 2.6, Proposition 3.19 implies that H is G -cr. In the general case HT is G -cr by the argument just given; thus H is G -cr by Theorem 3.10. \square

Our next result follows immediately from Proposition 2.15 and Theorem 3.1.

Corollary 3.21. *Let S be a linearly reductive group acting on the reductive group G by automorphisms and let $H = C_G(S)^0$. Suppose K is a closed subgroup of H . Then K is H -completely reducible if and only if it is G -completely reducible.*

Note that $H = C_G(S)^0$ is reductive, thanks to [32, Prop. 10.1.5]: in fact, by Corollary 3.21, H is G -cr, because H is H -cr. In the examples below we indicate how Corollary 3.21 leads to new criteria for completely reducible subgroups.

The following special case of Corollary 3.21, when S is a subtorus of G so that $H = C_G(S)$ is a Levi subgroup of G , is a result due to J.-P. Serre [38, Prop. 3.2]. This is again part of the philosophy mentioned in the Introduction, because for $G = \mathrm{GL}(V)$, the statement simply reduces to the fact that V is a semisimple K -module if and only if every K -submodule of V is semisimple.

Corollary 3.22. *Let K be a closed subgroup of a Levi subgroup L of G . Then K is L -completely reducible if and only if K is G -completely reducible.*

Other typical applications of Corollary 3.21 are when S is the group generated by a graph automorphism of G , or S is the group generated by a semisimple element of G such that $C_G(S)^0$ is a subgroup of G of maximal semisimple rank. The subsystems corresponding to maximal semisimple rank subgroups of a simple group G are determined by means of the algorithm of Borel and de Siebenthal [3], see also [6, Ex. Ch. VI §4.4]. We give some examples.

Example 3.23. Suppose that $p \neq 2$. Let V be a finite-dimensional k -vector space. Let H be either $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ and let K be a closed subgroup of H . Observe that H is the fixed point subgroup of an involution of $\mathrm{GL}(V)$ and thus Corollary 3.21 applies. Then K is H -completely reducible if and only if it is $\mathrm{GL}(V)$ -completely reducible, that is, if and only if V is a semisimple K -module; see also [33, Cor. 16.10], and [38, Ex. 3.2.2(b)].

Example 3.24. Suppose that $p > 3$. Let $G = \mathrm{SO}_8(k)$ and let H be the subgroup of G of type G_2 that is the fixed point subgroup of G under the triality graph automorphism of G . Let K be a closed subgroup of H . Then K is H -completely reducible if and only if it is G -completely reducible. By our assumption on p the natural 8-dimensional module $V_G(8)$ of G splits as an H -module into a direct sum $V_G(8)|_H = V_H(7) \oplus k$ of the simple 7-dimensional H -module $V_H(7)$ and the trivial module. It follows from Corollary 3.21 and Example 3.23 that K is H -completely reducible if and only if $V_H(7)$ is a semisimple K -module. We observe that this equivalence does also hold in characteristic 3, see [38, Ex. 3.2.2(c)].

Example 3.25. Let G be of type E_8 and suppose that $p \neq 5$. Let H be a maximal rank subgroup of G of type A_4A_4 . Let K be a closed subgroup of H . Then K is H -completely reducible if and only if it is G -completely reducible. This follows from Corollary 3.21, as H is the centralizer in G of an element of order 5. Using Lemma 2.12 this can then be interpreted in terms of the semisimplicity of the natural modules of the A_4 -factors of H for the corresponding projections of K into these factors.

More generally, we have the following result.

Theorem 3.26. *Suppose that p is good for G . Let H be a regular reductive subgroup of G . Let K be a closed subgroup of H . Then K is H -completely reducible if and only if K is G -completely reducible.*

Proof. Let T be a maximal torus that normalizes H . There exists a subtorus S of T such that S centralizes H , $H \cap S$ is finite and $T \subseteq HS$. Applying Lemma 2.12(i) and (ii) to the product map $H \times S \rightarrow HS$, we see that a closed subgroup K of H is H -cr if and only if it is HS -cr. Thus we may assume that H contains T .

As H is a regular reductive subgroup of G , it is G -cr, by Proposition 3.20. Thanks to Corollaries 3.5 and 3.22, we may assume that H is G -ir. By Remark 2.9, we may assume

that H is topologically finitely generated. Then, since $C_G(H)^0 = Z(G)^0$ by Proposition 2.13, we see that H has maximal semisimple rank. Finally, the result follows from the algorithm of Borel and de Siebenthal ([3] or [6, Ex. Ch. VI §4.4]), Carter's criterion [9, Prop. 11], and Corollary 3.21. \square

3.5. G -complete reducibility, separability and reductive pairs. Now we consider the interaction of subgroups of G with the Lie algebra $\text{Lie } G = \mathfrak{g}$ of G .

Definition 3.27. We say a closed subgroup H of G is *separable in G* if the Lie algebra centralizer $\mathfrak{c}_{\mathfrak{g}}(H)$ of H equals the Lie algebra of $C_G(H)$ (that is, if the scheme-theoretic centralizer of H in G is smooth). If the former properly contains the latter, then we say that H is *non-separable in G* .

Example 3.28. Any closed subgroup H of $G = \text{GL}(V)$ is separable in G . For separability means precisely that the centralizers of H in $\text{GL}(V)$ and in $\text{Lie GL}(V)$ have the same dimension, and for $\text{GL}(V)$ this holds, because the centralizer of H in $\text{GL}(V)$ is the open subset of invertible elements of the centralizer of H in $\text{Lie GL}(V) \cong \text{End } V$.

Example 3.29. (Cf. [20, Ex. 3.4(b)]). Let $p = 2$, $G = \text{PGL}_2(k)$ and let T be a maximal torus of G . Then $N_G(T)$ is non-separable in G .

Remark 3.30. If G itself is non-separable in G , there exists a reductive group \widehat{G} such that \widehat{G} is generated by G and a central torus, and \widehat{G} is separable in \widehat{G} , see [20, Thm. 4.5]. For example, $G = \text{SL}_p(k)$ is non-separable in G , and we can take $\widehat{G} = \text{GL}_p(k)$.

Remark 3.31. The terminology in Definition 3.27 is motivated as follows. Suppose that H is topologically generated by x_1, \dots, x_n in G . Then the orbit map $G \rightarrow G \cdot (x_1, \dots, x_n)$ is separable if and only if

$$\mathfrak{c}_{\mathfrak{g}}(H) = \mathfrak{c}_{\mathfrak{g}}(\{x_1, \dots, x_n\}) = \text{Lie } C_G((x_1, \dots, x_n)) = \text{Lie } C_G(H)$$

(cf. [2, Prop. 6.7]), i.e., if and only if H is separable in G .

Definition 3.32. Following Richardson [30], we call (G, H) a *reductive pair* provided H is a closed reductive subgroup of G and $\text{Lie } G$ decomposes as an H -module into a direct sum

$$\text{Lie } G = \text{Lie } H \oplus \mathfrak{m},$$

where H acts via the adjoint action Ad_G .

For a list of examples of reductive pairs we refer to P. Slodowy's article [40, I.3]. The next example gives a further class of reductive pairs.

Example 3.33. Let H be a reductive subgroup of G containing a maximal torus T of G , so that in particular, H is regular reductive. One can show that if p does not divide any of the structure constants of the Chevalley commutator relations of G , then (G, H) is a reductive pair. The complement to $\text{Lie } H$ in $\text{Lie } G$ is the sum of the root spaces of $\text{Lie } G$ corresponding to the roots in $\Psi(G, T)$ that lie outside $\Psi(H, T)$.

This is no longer valid if we relax the restriction on p . For instance, suppose $p = 2$ and let G be of type B_2 . If H is the regular reductive subgroup of type A_1^2 generated by the short root subgroups of G , then $\text{Lie } H$ does not admit an H -stable complement in $\text{Lie } G$.

The following observation, due to Serre, gives many more examples of reductive pairs.

Remark 3.34. Let $f : G_1 \rightarrow G_2$ be a homomorphism of reductive groups and let $df : \text{Lie } G_1 \rightarrow \text{Lie } G_2$ be the induced homomorphism on the Lie algebras. Suppose that there exists a symmetric Ad-invariant bilinear form (“Ad-invariant form” for short) $(\cdot, \cdot)_2$ on $\text{Lie } G_2$ which is non-degenerate. We then define an Ad-invariant form on $\text{Lie } G_1$ via $(x, y)_f := (df(x), df(y))_2$ for $x, y \in \text{Lie } G_1$. If $(\cdot, \cdot)_f$ is non-degenerate and $df : \text{Lie } G_1 \rightarrow \text{Lie } f(G_1)$ is surjective, then $(G_2, f(G_1))$ is a reductive pair. To see this, we take \mathfrak{m} to be the orthogonal complement of $\text{Lie } f(G_1)$ in $\text{Lie } G_2$ with respect to $(\cdot, \cdot)_2$, an $f(G_1)$ -stable subspace of $\text{Lie } G_2$. Our hypotheses imply that the restriction of $(\cdot, \cdot)_2$ to $\text{Lie } f(G_1)$ is non-degenerate, so $\text{Lie } f(G_1) \cap \mathfrak{m} = \{0\}$. As $(\cdot, \cdot)_2$ is non-degenerate, we have $\dim \mathfrak{m} + \dim f(G_1) = \dim G_2$, as required.

Suppose that $\text{Lie } G_1$ is simple and admits a non-degenerate Ad-invariant form $(\cdot, \cdot)_1$. It can be shown that $(\cdot, \cdot)_1$ is the unique Ad-invariant form up to scalar multiplication, so we have $(\cdot, \cdot)_f = \delta_f(\cdot, \cdot)_1$ for some $\delta_f \in k$. Thus $(G_2, f(G_1))$ is a reductive pair as long as $\delta_f \neq 0$. In analogy to [11, §2] we call δ_f the *Dynkin index* of f . For tables of the Dynkin index of the fundamental representations see [19, §5], where this invariant is called the “second index”; here $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ are fixed by appropriate normalization conditions.

Theorem 3.35. *Suppose that (G, H) is a reductive pair. Let K be a closed subgroup of H such that K is a separable subgroup of G . If K is G -completely reducible, then it is also H -completely reducible.*

Proof. By Remark 2.9, we can assume that K is topologically finitely generated, say by k_1, \dots, k_n . Let C be the G -orbit of (k_1, \dots, k_n) in G^n . By assumption, the orbit map $G \rightarrow C$ is separable, cf. Remark 3.31. Thanks to a generalization of a standard tangent space argument of Richardson [30, Thm. 4.1] to this situation, see Slodowy [40, Thm. 1], the intersection $C \cap H^n$ is a finite union of H -conjugacy classes, each of which is closed in $C \cap H^n$; the second of these assertions follows directly from the proof in loc. cit., as each irreducible component of $C \cap H^n$ is a single H -orbit.

Now suppose that K is G -cr. Then C is closed in G^n by Corollary 3.7, and so the H -orbit of (k_1, \dots, k_n) is closed in H^n by the above argument. Thus, using Corollary 3.7 again, we see that K is H -cr, as desired. \square

Example 3.28 shows that the separability hypothesis is automatically satisfied for the case $G = \text{GL}(V)$. We obtain an immediate consequence of Theorem 3.35, which is in the spirit of Serre’s Theorem 3.41(ii) below.

Corollary 3.36. *Suppose that $(\text{GL}(V), H)$ is a reductive pair and K is a closed subgroup of H . If V is a semisimple K -module, then K is H -completely reducible.*

In our next example we look at the special case of the adjoint representation.

Example 3.37. Let H be a simple group of adjoint type and let $G = \text{GL}(\text{Lie } H)$. We have a symmetric non-degenerate Ad-invariant bilinear form on $\text{Lie } G \cong \text{End}(\text{Lie } H)$ given by the usual trace form and its restriction to $\text{Lie } H$ is just the Killing form of $\text{Lie } H$. Since H is adjoint and Ad is a closed embedding, $\text{ad} : \text{Lie } H \rightarrow \text{Lie } \text{Ad}(H)$ is surjective. Thus it follows from the arguments in the first paragraph of Remark 3.34 that if the Killing form of $\text{Lie } H$ is non-degenerate, then (G, H) is a reductive pair.

Suppose first that H is a simple classical group of adjoint type and $p > 2$. The Killing form is non-degenerate for $\mathfrak{sl}(V)$, $\mathfrak{so}(V)$, or $\mathfrak{sp}(V)$ if and only if p does not divide $2 \dim V$, $\dim V - 2$, or $\dim V + 2$, respectively, cf. [7, Ex. Ch. VIII, §13.12]. In particular, for H

adjoint of type A_n , B_n , C_n , or D_n , the Killing form is non-degenerate if $p > 2$ and p does not divide $n + 1$, $2n - 1$, $n + 1$, or $n - 1$, respectively.

Now suppose that H is a simple exceptional group of adjoint type. If p is good for H , then the Killing form of $\text{Lie } H$ is non-degenerate; this was noted by Richardson (see [30, §5]). Thus if p satisfies the appropriate condition, then $(\text{GL}(\text{Lie } H), H)$ is a reductive pair and Corollary 3.36 applies.

3.6. G -ir subgroups. We can say more about the centralizers of G -irreducible subgroups than in Corollary 3.17:

Lemma 3.38. *Let H be a G -irreducible subgroup of G . Then $C_G(H)$ is linearly reductive.*

Proof. Suppose that u is a non-trivial unipotent element of $C_G(H)$. Then H centralizes the non-trivial unipotent subgroup generated by u . By a construction due to Borel and Tits [21, Prop. 5.4(b)], H is not G -irreducible, a contradiction. We conclude that all elements of $C_G(H)$ are semisimple, and we deduce from [28, §4, Thm. 2] that $C_G(H)$ is linearly reductive. \square

In [16, Thm. 3] M. Liebeck and G. Seitz proved that if G is simple of exceptional type, H is a simple subgroup of G , and $p > 7$, then H is a separable subgroup of G . The next result shows that even in low characteristic there is not much freedom for a G -ir subgroup to be non-separable in G .

Proposition 3.39. *Let $H \subseteq G$ be a non-separable G -irreducible subgroup of G . Then there exists a regular non-separable subgroup M of G containing H with M^0 reductive.*

Proof. Let $x \in \mathfrak{c}_g(H)$ such that x is not in the Lie algebra of $C_G(H)$, and set $M = C_G(x)$. Since $H \subseteq M$ and H is G -ir, M is G -ir; in particular, M^0 is reductive. Thanks to [20, Lem. 2.4], x is semisimple, so M contains a maximal torus of G , which implies that M is regular. Since $C_G(M) \subseteq C_G(H)$, we see that x is not in the Lie algebra of $C_G(M)$, but by definition of M we have $x \in \mathfrak{c}_g(M)$, so M is non-separable in G . \square

The following result answers a question raised by M. Liebeck and D. Testerman [18].

Proposition 3.40. *Suppose that H is a (non-connected) G -irreducible subgroup of G where H^0 is not G -irreducible. Then $C_G(H^0)$ contains a non-central torus of G .*

Proof. As H is G -ir, it is G -cr, so H^0 is G -cr, by Theorem 3.10. By hypothesis, H^0 is not G -ir, so there exists a proper parabolic subgroup of G containing H^0 . As H^0 is G -cr, it lies in a Levi subgroup of this parabolic subgroup. This Levi subgroup is the centralizer in G of some non-central torus S of G (since the Levi subgroup is proper in G). Thus $C_G(H^0)$ contains S , as desired. \square

3.7. G -complete reducibility and semisimple modules. There is a fundamental connection between Serre's notion of G -complete reducibility and the semisimplicity of G -modules. Let h denote the Coxeter number of G . For a finite-dimensional G -module V define $n(V) = \max\{\sum_{\alpha > 0} \langle \lambda, \alpha^\vee \rangle\}$, where the maximum is taken over all T -weights λ of V . Observe that if V is *non-degenerate*, i.e., when the connected kernel of the representation of G on V is a torus, then $n(V) \geq h - 1$, cf. [37, p. 20]. The following is the main objective in [37], see also [38, Thm. 5.4].

Theorem 3.41. *Let V be a G -module with $p > n(V)$. Let H be a closed subgroup of G .*

- (i) *If H is G -completely reducible, then V is a semisimple H -module. In particular, V is a semisimple G -module.*
- (ii) *If V is non-degenerate and semisimple as an H -module, then H is G -completely reducible.*

This is a deep theorem and its proof, which is quite involved, requires Serre's notion of saturation, cf. [37, p. 22], [38, §5]. The proof of part (ii) uses the full force of Theorem 3.48 below, which itself is a very difficult case-by-case analysis.

The following is the special case of Theorem 3.41 for the adjoint module, [38, Cor. 5.5]. Note that $n(\mathfrak{g}) = 2h - 2$ and \mathfrak{g} is a non-degenerate G -module.

Corollary 3.42. *Let H be a closed subgroup of G . Suppose that $p > 2h - 2$. Then $\text{Lie } G$ is a semisimple H -module if and only if H is G -completely reducible.*

Remarks 3.43. (i). For $G = \text{GL}(V)$ the forward implication in Corollary 3.42 does not require any restrictions on p (see [35, Thm. 3.3]; this is also a special case of Theorem 3.46 below, cf. Example 3.28).

(ii). It follows from Corollary 3.36 and Example 3.37 that for the forward implication in Corollary 3.42, it suffices to require that $p > 2$ and p does not divide $n + 1, 2n - 1, n + 1, n - 1$, in case G is an adjoint simple group of type A_n, B_n, C_n, D_n , respectively, and that p is good for G in case G is an adjoint simple group of exceptional type. For example, if G is adjoint of type C_n , then it suffices to require that $p > 2$ and p does not divide $n + 1$, which improves on the bound $p > 2h - 2 = 4n - 2$ from Corollary 3.42; if G is of type E_8 , then we obtain the bound $p > 5$, which improves on the bound $p > 2h - 2 = 58$ from Corollary 3.42.

(iii). It is shown in [17, Cor. 3] that if G is simple of exceptional type and H is simple of rank at least 2, then it suffices to require that $p > 7$ for the reverse implication of Corollary 3.42 to hold. A computation of J.-P. Serre shows (except possibly when $G = G_2$ and $p = 5$ or $G = B_3$ and $p = 3$) that if $p \leq 2h - 2$, then there always exists a subgroup H of G of type A_1 which is G -cr, while $\text{Lie } G$ is not H -semisimple, cf. [38, Rem. 5.6]. In particular, this says that the bound in Corollary 3.42 is sharp for the reverse implication.

To go with the counterexamples to the reverse implication of Corollary 3.42 when $p \leq 2h - 2$ referred to in Remark 3.43(iii), here is another example of the failure of the conclusion of Theorem 3.41(i) if $p \leq n(V)$.

Example 3.44. Let $p > 2$. The adjoint representation of $H := \text{PGL}_p(k)$ on \mathfrak{h} has a composition factor of dimension $m := p^2 - 2$. So we can regard H as an $\text{SL}_m(k)$ -irreducible subgroup of $\text{SL}_m(k)$. The image of H in $G := \text{PGL}_m(k)$ is G -irreducible, by Lemma 2.12(ii)(a), and it is clear that this image is isomorphic to H . So we can regard H as a G -irreducible subgroup of G . Now \mathfrak{g} is simple as a G -module, as p is coprime to m . However, \mathfrak{g} is not semisimple as an H -module: for the H -submodule \mathfrak{h} of \mathfrak{g} is not semisimple. Note that $n(\mathfrak{g}) = 2m - 2$.

The following example, due to M.W. Liebeck, shows that Theorem 3.41(ii) fails without the restriction on p .

Example 3.45. Suppose that $p = 2, m \geq 4$ is even and $H := \text{Sp}_m(k)$ is embedded diagonally in the maximal rank subgroup $M := \text{Sp}_m(k) \times \text{Sp}_m(k)$ of $G := \text{Sp}_{2m}(k)$. Let V and V' be the natural modules for the $\text{Sp}_m(k)$ -factors of M , so that the orthogonal direct sum $W := V \oplus V'$ is the natural module for G . Then, as V and V' are irreducible $\text{Sp}_m(k)$ -modules, W is a

semisimple H -module. Choose a form-preserving $\mathrm{Sp}_m(k)$ -module isomorphism $f : V \rightarrow V'$. It is easily checked that the only m -dimensional H -stable subspaces of W are V' and the subspaces V_a defined by $V_a := \{v + af(v) \mid v \in V\}$, where $a \in k$. Since $(v_1 + af(v_1), v_2 + af(v_2)) = (1 + a^2)(v_1, v_2)$ for every $v_1, v_2 \in V$, we see that V_1 is the unique H -stable, totally singular m -dimensional subspace of W . This implies that H lies in a proper parabolic subgroup P of G , but H does not lie in any Levi subgroup of P . Thus H is not G -cr.

On the other hand, it follows from Lemma 2.12(i) (applied to the diagonal embedding $H \rightarrow M$) that H is M -cr, and further from Corollary 3.21 that if $p \neq 2$, then H is G -cr, as the maximal rank subgroup M is the centralizer of an involution in G . This improves on the bound $p > n(W) = 2m = h$ from Theorem 3.41(ii) in this particular case.

Our next result gives another sufficient condition for the forward direction of Corollary 3.42 to hold; recall Definition 3.27 of a separable subgroup of G .

Theorem 3.46. *Let H be a separable subgroup of G . If \mathfrak{g} is semisimple as an H -module, then H is G -completely reducible.*

Proof. By Remark 2.9, we can assume that H is topologically finitely generated, say by h_1, \dots, h_n . Suppose that H is not G -cr. Then, by Corollary 3.7, the G -orbit of (h_1, \dots, h_n) is not closed in G^n . By the Hilbert–Mumford Theorem 2.3 this implies that there exists a cocharacter λ of G such that $\lim_{t \rightarrow 0} \lambda(t) \cdot (h_1, \dots, h_n)$ exists — call this limit (h'_1, \dots, h'_n) — and such that $G \cdot (h'_1, \dots, h'_n)$ is closed. In particular, we have

$$(3.47) \quad \dim(G \cdot (h'_1, \dots, h'_n)) < \dim(G \cdot (h_1, \dots, h_n)).$$

Let H' be the algebraic group generated by h'_1, \dots, h'_n , so that $H' = c_\lambda(H)$, where c_λ is the map defined in Lemma 2.4(iii). Note that $H \subseteq P_\lambda$, since $\lim_{t \rightarrow 0} \lambda(t) \cdot h_i$ exists for each $i = 1, \dots, n$. Inequality (3.47) implies that $\dim C_G(H') > \dim C_G(H)$. Since H is separable in G , we deduce that $\dim \mathfrak{c}_\mathfrak{g}(H') > \dim \mathfrak{c}_\mathfrak{g}(H)$.

Let $M = \mathrm{Ad}_G(H)$ and $M' = \mathrm{Ad}_G(H')$. It is clear that $M' = c_{\mathrm{Ad}_G \circ \lambda}(M)$. Since \mathfrak{g} is H -semisimple, M is $\mathrm{GL}(\mathfrak{g})$ -cr. It follows from Lemma 2.17 and Theorem 3.1 that M' is $\mathrm{GL}(\mathfrak{g})$ -conjugate to M . Now $\mathfrak{c}_\mathfrak{g}(H)$ (respectively $\mathfrak{c}_\mathfrak{g}(H')$) is precisely the set of fixed points of M (respectively M') in \mathfrak{g} , so $\mathfrak{c}_\mathfrak{g}(H')$ is $\mathrm{GL}(\mathfrak{g})$ -conjugate to $\mathfrak{c}_\mathfrak{g}(H)$. But this implies that $\dim \mathfrak{c}_\mathfrak{g}(H') = \dim \mathfrak{c}_\mathfrak{g}(H)$, a contradiction. We conclude that H is G -cr, as required. \square

The following is a simplified statement of a consequence of a number of deep theorems due to J.C. Jantzen [12] and G. McNinch [23] in case G is classical and M. Liebeck and G. Seitz [16] for G of exceptional type; see [38, Thm. 4.4] and [36, §3].

Theorem 3.48. *Let H be a closed connected subgroup of G and suppose that $p > h$. Then H is G -completely reducible if and only if H is reductive.*

Theorem 3.48 says that provided $p > h$, for a connected subgroup of G the notions of reductivity and G -complete reducibility are equivalent, as in characteristic zero, cf. Subsection 2.2. As the results in [16] and [23] depend on case-by-case studies, it would be desirable to have a uniform proof of Theorems 3.41 and 3.48, even with some additional restrictions on p . We believe that Theorem 3.35, Corollary 3.36, and Theorem 3.46 provide a first step in this direction.

If $(\mathrm{GL}(V), H)$ is a reductive pair, then V is non-degenerate by definition. The dependence on the characteristic in Corollary 3.36 is buried in the hypothesis that $(\mathrm{GL}(V), H)$ is a

reductive pair, see Example 3.33, Remark 3.34, and Example 3.37. The advantages of Theorem 3.35, Corollary 3.36, and Theorem 3.46 are that they do not require the notion of saturation and they are free of case-by-case considerations.

4. G -COMPLETE REDUCIBILITY AND BUILDINGS

In this section we consider the connection between the notion of G -complete reducibility and the building of G due to J.-P. Serre [36, Thm. 2]. For an arbitrary spherical building X , a subset Y of X is said to be *convex* if whenever two points of Y are not opposite in X , then Y contains the unique geodesic joining these points. A convex subset Y is *X -completely reducible* (X -cr) if for every $y \in Y$ there exists a point $y' \in Y$ opposite to y in X , [38, Def. 2.2.1]. The vertices of X can be labelled in an essentially unique way via an equivalence relation on vertices, cf. [8, p30], [38, 2.1.2]; the *type* of a vertex is its label. An automorphism f of X is said to be *type-preserving* if x and $f(x)$ have the same type for all vertices x of X .

Now let $X = X(G)$ be the spherical Tits building of G , cf. [8], [43]. Recall that the simplices in X correspond to the parabolic subgroups of G and the vertices of X correspond to the maximal proper parabolic subgroups, see [38, §3.1]. For a subgroup H of G let X^H be the fixed point subcomplex of the action of H , i.e., the subcomplex of all H -stable (thus H -fixed) simplices in X . This subcomplex is always convex; if it is also X -completely reducible, then we say H *acts completely reducibly on X* , [38, §2.3]. For any subgroup H of G the action of H on X is type-preserving.

Take a Levi subgroup L of G and let $s(L) := X^L$ denote the subcomplex of X consisting of the parabolic subgroups of G containing L . For every parabolic subgroup P in $s(L)$ there is a unique Levi subgroup M of P with $L \subseteq M$. Moreover, the parabolic subgroup P^- such that $P \cap P^- = M$ is also contained in $s(L)$, so that P has an opposite in $s(L)$; thus $s(L)$ is X -cr. This argument also shows that each P in $s(L)$ has a unique opposite in $s(L)$, and this implies that the geometric realization of $s(L)$ has the homotopy type of a single sphere (cf. Theorem 4.1(v) below). Serre calls the subcomplexes $s(L)$ of X *Levi spheres*, [36, §2] or [38, 2.1.6, 3.1.7]. The following is part of [36, Thm. 2] in our context.

Theorem 4.1. *Let H be a closed subgroup of G . Then the following are equivalent:*

- (i) H is G -completely reducible;
- (ii) X^H is X -completely reducible;
- (iii) X^H contains a Levi sphere of the same dimension as X^H ;
- (iv) X^H is not contractible (i.e., does not have the homotopy type of a point);
- (v) X^H has the homotopy type of a bouquet of spheres.

J.-P. Serre observed that Theorem 3.1 can also be interpreted in terms of Theorem 4.1. Note that any Levi sphere $s(L)$ contained in X^H is of the form $s(C_G(T))$, where T is a subtorus of $C_G(H)$. Thus $s(L)$ is of maximal dimension in X^H if and only if $L = C_G(S)$, where S is a maximal torus of $C_G(H)$. Moreover, $\dim X^H = \dim s(L)$ if and only if H is not contained in a proper parabolic subgroup of the Levi subgroup L . Thus part (iii) of Theorem 4.1 is equivalent to H being strongly reductive in G . We are grateful to J.-P. Serre for this building-theoretic interpretation.

Thanks to Theorem 4.1, the results of the previous section have counterparts in terms of buildings; e.g., Corollary 3.16 then says that for a closed subgroup H of G the fixed point subcomplex X^H is contractible if and only if $X^{N_G(H)}$ is contractible, etc.

If H is a G -cr subgroup of G , then it follows from Theorem 4.1 that X^H is itself a spherical chamber complex, much like a building. However, in general X^H is not a building. For instance, let T be a maximal torus in G . Then T is G -cr. However, X^T is a Coxeter complex and so is not a building, as X^T is not thick. (Recall that a chamber complex is said to be *thick* if every simplex of codimension one is contained in at least three chambers, e.g., see [8].) See also Example 4.5 below.

The following lemma extends a result that is well known when H is a torus.

Lemma 4.2. *Let H be a G -completely reducible subgroup of G and let P be a parabolic subgroup of G containing H . Then $P \cap C_G(H)^0$ is a parabolic subgroup of $C_G(H)^0$. Moreover, all parabolic subgroups of $C_G(H)^0$ arise in this way.*

Proof. First notice that since H is G -cr, $C_G(H)^0$ is reductive by Proposition 3.12. Since H is G -cr, there exists $\lambda \in Y(G)$ such that $H \subseteq L_\lambda(G) \subseteq P_\lambda(G) = P$ by Lemma 2.4. Then $\lambda(k^*) \subseteq C_G(H)^0$, so that $\lambda \in Y(C_G(H)^0)$ and $P \cap C_G(H)^0 = P_\lambda(C_G(H)^0)$ is a parabolic subgroup of $C_G(H)^0$, by Lemma 2.4, as claimed.

Now for any parabolic subgroup Q of $C_G(H)^0$, there exists $\mu \in Y(C_G(H)^0)$ such that $Q = P_\mu(C_G(H)^0)$. Then the parabolic subgroup $P = P_\mu(G)$ of G contains H , since $\mu(k^*)$ centralizes H , and $Q = P \cap C_G(H)^0$ by Corollary 2.5. \square

For an arbitrary reductive group K , let $X(K)$ denote the spherical Tits building of K . In our next result, Lemma 4.2 allows us to relate the fixed point complex $X(HC_G(H)^0)^H$ and the building of the connected centralizer of H in case H is a connected G -completely reducible subgroup of G .

Proposition 4.3. *Let H be a connected G -completely reducible subgroup of G . Then the chamber complexes $X(HC_G(H)^0)^H$ and $X(C_G(H)^0)$ are isomorphic. In particular, we see that $X(HC_G(H)^0)^H$ is itself a building.*

Proof. Set $M = HC_G(H)^0$ and observe that M is reductive by Proposition 3.12. Since H is normal in M , H is M -cr by Theorem 3.10. Also, since $C_G(H)^0 \subseteq M$, we have $C_M(H)^0 = C_G(H)^0$.

By Lemma 4.2 above, parabolic subgroups P of M containing H correspond bijectively to parabolic subgroups $P \cap C_M(H)^0$ of $C_M(H)^0 = C_G(H)^0$: to see that the correspondence is one-to-one, notice that if $H \subseteq P \subseteq M = HC_G(H)^0$, then we have $P = H(P \cap C_G(H)^0)$. Thus, for two parabolic subgroups P and Q of M containing H , we see that $P = Q$ if and only if $P \cap C_G(H)^0 = Q \cap C_G(H)^0$.

This bijection on the sets of parabolic subgroups affords the desired isomorphism on the underlying chamber complexes. \square

To illustrate Proposition 4.3 we first point to a trivial case:

Example 4.4. Let G be a reductive group and let H be a proper closed connected normal subgroup of G . Since H is normal in G , H is G -cr, by Theorem 3.10. As a special case of Proposition 4.3 we obtain that X^H is isomorphic to $X(C_G(H)^0)$.

The following example was communicated to us by B. Mühlherr.

Example 4.5. Let G be of type F_4 and let H be the simple A_1 -factor of the maximal rank subgroup M of G of type A_1C_3 . Then, by Proposition 3.20, H is G -cr, so that X^H does have

the homotopy type of a bouquet of spheres, by Theorem 4.1. However, X^H is not a building, as the apartments in X^H are not Coxeter complexes. Note that we have $HC_G(H) = M$. It follows from Proposition 4.3 that $X(M)^H$ is a building of type C_3 .

Group actions on buildings are also considered in [25] and [27] (Section 5 in [27] is closely related to Theorem 4.1). Given a spherical chamber complex, one can associate to it a building via a so-called *thickening* procedure, see [25, §1.7] and [27, §5] for details. It follows from [25, 1.7.26, 1.8.22, 3.4.8] that for a G -completely reducible subgroup H of G , the fixed point subcomplex X^H is a building if and only if it is thick. Moreover, for any G -completely reducible subgroup H of G , the thickening of X^H is isomorphic to $X(C_G(H)^0)$. Note that if X^H is already a building, then thickening has no effect. These results hold in greater generality than is stated here. For example, they are true for any group H which acts on G such that the induced action on X is completely reducible and type-preserving, see [25].

Now suppose that Y is a *strictly convex* subcomplex of X , i.e., Y is convex but it does not contain any two opposite points of X . Suppose the subgroup H of G stabilizes Y . The so-called “Center Conjecture” due to J. Tits claims the existence of a fixed point of H in Y , cf. [36, §4]. It turns out that Theorem 3.10 is a consequence of Tits’ Center Conjecture, [38, Prop. 2.11]. We refer to [38, §2.4] for more details and known instances of this conjecture; see also [25, §3.6] and [26] for related results.

5. RATIONALITY QUESTIONS

The notions of G -complete reducibility, etc., can be extended to reductive groups defined over arbitrary fields; see [38]. In this section k denotes an arbitrary field, not necessarily algebraically closed, \bar{k} denotes the algebraic closure of k , and G denotes a reductive group defined over k (see [2], [4], and [41] for more details). Given a field extension k'/k , we denote by $G(k')$ the group of k' -rational points of G ; if k' is algebraically closed, then we often identify G with $G(k')$. By a k' -subgroup H of G , we mean an algebraic subgroup of G over k' . We call a parabolic subgroup of G that is defined over k' a k' -parabolic subgroup of G . If P is a k' -parabolic subgroup of G and L is a Levi subgroup of P that is defined over k' , then we call L a k' -Levi subgroup of P .

Definition 5.1. Let k'/k be a field extension. We say that a k' -subgroup H of G is G -completely reducible over k' if whenever H is contained in a k' -parabolic subgroup P of G , there is a k' -Levi subgroup L of P such that $H \subseteq L$.

Remark 5.2. In particular, if k' is algebraically closed, then H is G -completely reducible over k' if and only if $H(k')$ is $G(k')$ -completely reducible as defined in Section 1.

5.1. Algebraically closed fields. First we consider extensions k'/k of algebraically closed fields. In view of Theorem 3.1, we record some results from [21], replacing “strongly reductive in G ” with “ G -cr”: parts (i) and (ii) of the following theorem are [21, Prop. 10.2] and [21, Thm. 10.3], respectively.

Theorem 5.3. *Let k'/k be an extension of algebraically closed fields.*

- (i) *Let H be a k -subgroup of G . Then H is G -completely reducible over k' if and only if H is G -completely reducible over k .*

(ii) Let K be a k' -subgroup of G such that K is G -completely reducible over k' . Then there exists a k -subgroup H of G such that H is G -completely reducible over k and H is $G(k')$ -conjugate to K .

Remark 5.4. Part (i) of Theorem 5.3 also holds replacing “completely reducible” with “irreducible” or “indecomposable”: for a closed subgroup is G -ind if and only if it is not centralized by any non-central torus of G and is G -ir if and only if it is both G -cr and G -ind.

If k'/k is an extension of algebraically closed fields, then two k -subgroups H_1 and H_2 of G are $G(k')$ -conjugate if and only if they are $G(k)$ -conjugate. This, together with Theorem 5.3, establishes the following result.

Corollary 5.5. *Let k'/k be an extension of algebraically closed fields. Then the map $H(k) \mapsto H(k')$ gives rise to a bijection between the set of $G(k)$ -conjugacy classes of k -subgroups of G that are G -completely reducible over k , and the set of $G(k')$ -conjugacy classes of k' -subgroups of G that are G -completely reducible over k' .*

Corollary 5.6. *Suppose that k is algebraically closed. Then there are only countably many conjugacy classes of k -subgroups of G that are G -completely reducible over k .*

Proof. Let k_0 be the algebraic closure of the prime field of k and let k_1 be the algebraic closure of $k_0(t)$, where t is transcendental over k_0 . The group G admits a k_0 -structure by [21, Prop. 3.2], so we can assume that G is defined over k_0 . By Corollary 5.5, we can assume that $k = k_1$; in particular, k is countable and k/k_0 is transcendental. Now any k -subgroup of G that is G -cr over k is reductive, and hence by [21, Lem. 9.2] is topologically finitely generated. But $G(k)$, being countable, has only countably many topologically finitely generated subgroups, so the result follows. \square

Remark 5.7. We cannot replace “countably many” by “finitely many” in the previous corollary: see Remark 3.9(ii).

5.2. Perfect fields. Now we consider field extensions k'/k where both k' and k are perfect. If k is perfect, k'/k is a Galois extension, and X is a variety defined over k , then a k' -subvariety of X is defined over k if and only if it is $\text{Gal}(k'/k)$ -stable, cf. [2, AG §14]. The forward implication of the following result uses the argument of [15, Prop. 2.2], but without the extra complication that appears there of graph automorphisms; [15, Prop. 2.2] relies on a version of the Hilbert–Mumford–Kempf Theorem (see [14, Thm. 4.2]) which requires k to be perfect. The reverse implication of Theorem 5.8 is an observation of J.-P. Serre.

Theorem 5.8. *Let k'/k be an extension of perfect fields and let H be a k -subgroup of G . Then H is G -completely reducible over k' if and only if H is G -completely reducible over k .*

Proof. It suffices to prove the result in the special case when k' is algebraically closed. By Theorem 5.3(i), we can assume that $k' = \bar{k}$. Suppose that H is not G -cr over \bar{k} . By Lemma 2.10, there exists a finitely generated subgroup Γ of $H(\bar{k})$ such that for every \bar{k} -parabolic subgroup P of G and every \bar{k} -Levi subgroup L of a \bar{k} -parabolic subgroup of G , we have $\Gamma \subseteq P(\bar{k})$ if and only if $H \subseteq P$ and $\Gamma \subseteq L(\bar{k})$ if and only if $H \subseteq L$.

As k is separable and \bar{k}/k is algebraic, we can choose a finite Galois extension k_1/k such that $\Gamma \subseteq H(k_1)$. Let Γ_1 be the group generated by the $\text{Gal}(\bar{k}/k)$ -conjugates of Γ and let M be the closure of Γ_1 in H . Then Γ_1 is a finitely generated $\text{Gal}(\bar{k}/k)$ -stable subgroup of $G(\bar{k})$,

and we can choose a finite set of generators h_1, \dots, h_n for Γ_1 such that the h_i are permuted by $\text{Gal}(\bar{k}/k)$. By Theorem 3.1 and the argument of [15] applied to the tuple (h_1, \dots, h_n) (see the proof of [15, Prop. 2.2] and the paragraph that follows it), there exists a $\text{Gal}(\bar{k}/k)$ -stable \bar{k} -parabolic subgroup P of G such that $M \subseteq P$ but M does not lie in any \bar{k} -Levi subgroup of P . Then P is defined over k , $H \subseteq P$ and H does not lie in any \bar{k} -Levi subgroup of P ; in particular, H does not lie in any k -Levi subgroup of P . It follows that H is not G -cr over k .

Conversely, suppose that H is G -cr over \bar{k} . Let Q be a k -parabolic subgroup of G such that $H \subseteq Q$. The centralizer $C_Q(H)$ is defined over k , because H and Q are, so by [2, Thm. 18.2], $C_Q(H)$ contains a maximal torus S defined over k . As H is G -cr over \bar{k} , H is contained in a \bar{k} -Levi subgroup M of Q . Conjugating M by some element of $C_Q(H)$ if necessary, we can assume that the torus $Z(M)^0$ is contained in S . We have $C_G(S) \subseteq C_G(Z(M)^0) = M$. Now $C_G(S)$ is defined over k , because S is, so $C_G(S)$ contains a maximal torus T defined over k , again by [2, Thm. 18.2]. There is exactly one Levi subgroup of Q containing any given maximal torus, cf. [41, Cor. 8.4.4], so it follows that M is defined over k . Thus H is G -cr over k . \square

Remark 5.9. One can extend the definition of G -ir and G -ind to the non-algebraically closed setting in the obvious way, but the analogue of Theorem 5.8 does not carry over (cf. Remark 5.4); for example, take k perfect but not algebraically closed and consider a k -subgroup H of $G = \text{GL}_n(k)$ that is irreducible but not absolutely irreducible.

The following special case has applications to finite groups of Lie type (cf. [15, Prop. 2.2]).

Example 5.10. Suppose that G is defined over the finite field \mathbb{F}_p , let k' be the algebraic closure of \mathbb{F}_p , and let $\sigma : G(k') \rightarrow G(k')$ be some power of Frobenius. Let G_σ be the subgroup of fixed points of σ . Then any subgroup F of G_σ either is G -completely reducible over k' , or is contained in a proper σ -stable parabolic subgroup of G . To see this, observe that we have $G_\sigma = G(k)$ for some finite extension k of \mathbb{F}_p . We can regard F as a k -subgroup of G . If F is not G -cr over k' , then F is not G -cr over k , by Theorem 5.8, so there is a k -parabolic subgroup P of G such that F is not contained in any k -Levi subgroup of P . In particular, P is a proper σ -stable parabolic subgroup of G , as required. If P is chosen as in the proof of Theorem 5.8, then in fact F is not contained in any k' -Levi subgroup of P at all.

We are grateful to G. McNinch for the following example, which shows that the reverse implication in Theorem 5.8 fails if the hypothesis of perfection is removed.

Example 5.11. Let k_1/k be a purely inseparable field extension of degree p . Set $k' = \bar{k}$. We can regard k_1^* as an algebraic group H over k ; the action of k_1^* on k_1 by left multiplication gives rise to a k -embedding of H in $G = \text{GL}(k_1)$, where we regard k_1 as a k -vector space. As H acts transitively on k_1^* , H cannot stabilize any proper non-trivial k -subspace of k_1 , so H is G -cr over k . However, H is not reductive: for the homomorphism ϕ sending x to x^p maps H onto the group of scalar multiples of the identity matrix, so $\ker \phi$ is a $(p-1)$ -dimensional normal unipotent subgroup of H . This implies that H is not G -cr over k' .

6. THE NON-CONNECTED CASE

In this section we extend Theorem 3.1 and many of our other results to groups G that are no longer required to be connected. The formalism we use here for dealing with such groups is taken from [21], which in turn is based on the approaches of Vinberg [45] and Richardson

[33]. Thus for the remainder of the paper we suppose that G is a linear algebraic group with G^0 reductive; we call this “the non-connected case” and such a group is referred to as a “non-connected reductive group”; note, however, that we do not exclude the case $G = G^0$.

The idea is to use the appropriate generalization of the notion of a parabolic subgroup to the non-connected case, using the formalism of Lemma 2.4 (cf. [21, §5]). For $\lambda \in Y(G)$, we call a subgroup of the form $P_\lambda := \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$ a *Richardson parabolic* (or *R-parabolic*) subgroup of G (cf. [33, §2]; in [21], these were called generalized parabolic subgroups). By [41, Lem. 6.2.4], any R-parabolic subgroup P of G is a parabolic subgroup of G in the sense that G/P is a complete variety, but the converse is false, cf. [21, Rem. 5.3]. We call a subgroup of the form $L_\lambda := C_G(\lambda(k^*))$ a *Richardson Levi* (or *R-Levi*) subgroup of P_λ . By an R-Levi subgroup of G , we mean an R-Levi subgroup of some R-parabolic subgroup of G . We have $P_\lambda = L_\lambda \ltimes R_u(P_\lambda)$. If L is an R-Levi subgroup of G , then $L = C_G(Z(L)^0)$ and $C_G(L) = Z(L)$.

We define G -complete reducibility, G -irreducibility and G -indecomposability as we did in the connected case, replacing parabolic subgroups and Levi subgroups with R-parabolic subgroups and R-Levi subgroups respectively, and we extend the definition of strong reducitivity in G similarly. If $\lambda \in Y(G)$, then $P_\lambda^0 = P_\lambda(G^0)$ is a parabolic subgroup of G^0 and $L_\lambda^0 = L_\lambda(G^0)$ is a Levi subgroup of P_λ^0 . Conversely, if P is any parabolic subgroup of G^0 , then $P = P_\lambda^0$ for some $\lambda \in Y(G)$; moreover, if L is a Levi subgroup of P , then λ can be chosen so that $L = L_\lambda^0$. Thus the above definitions of G -complete reducibility, etc., agree with those from Section 1 if G is connected.

The proof of the non-connected version of Theorem 3.1 is given in Subsection 6.3. We use the theorem to prove several of the results (e.g., the non-connected version of Proposition 2.15) in Subsections 6.1 and 6.2; the proof of the non-connected version of Theorem 3.1 does not depend on these results.

6.1. Preliminaries. In order to generalize our work to the non-connected case, we need to prove the analogues of a number of results which are standard when G is connected. Chief among these is [4, Prop. 4.4] which is central to the proof of Theorem 3.1.

First notice that all of Lemma 2.4 extends to the non-connected case, replacing parabolic subgroups and Levi subgroups with R-parabolic subgroups and R-Levi subgroups respectively. It is clear that $R_u(P_\lambda) = \ker c_\lambda$ is connected, so $R_u(P_\lambda) = R_u(P_\lambda^0)$; in particular, L_λ meets every component of P_λ . For more details, see [21, §5].

The next result is [21, Prop. 5.4(a)].

Proposition 6.1. *Let P be a parabolic subgroup of G^0 . Then $N_G(P)$ is an R-parabolic subgroup of G .*

Note that $N_G(P)^0 = P$, since a parabolic subgroup of a connected reductive group is self-normalizing.

Part (ii) of the next lemma provides the extension of [4, Prop. 4.4(c)]. Notice that part (iii) is also standard for connected G .

Lemma 6.2. (i) *Let $\lambda, \mu \in Y(G)$ such that $\lambda(k^*)$ and $\mu(k^*)$ commute. Then for all sufficiently large $m \in \mathbb{N}$, we have $P_{m\lambda+\mu} = P_\mu(L_\lambda) \ltimes R_u(P_\lambda)$ and $L_{m\lambda+\mu} = L_\mu(L_\lambda)$. In particular, $P_{m\lambda+\mu} \subseteq P_\lambda$.*

- (ii) If P is an R -parabolic subgroup of G and L is an R -Levi subgroup of P , then the R -parabolic subgroups of G contained in P are precisely the subgroups of the form $P' \ltimes R_u(P)$ with P' an R -parabolic subgroup of L .
- (iii) If P and Q are R -parabolic subgroups of G with R -Levi subgroups L and M respectively, such that $L \cap M$ contains a maximal torus T of G , then

$$(6.3) \quad P \cap Q = (L \cap M)(L \cap R_u(Q))(R_u(P) \cap M)(R_u(P) \cap R_u(Q)),$$

and $R_u(P \cap Q)$ is the product of the last three factors.

Proof. (i). The inclusions $P_{m\lambda+\mu} \subseteq P_\lambda$ and $L_{m\lambda+\mu} \subseteq L_\lambda$ for large m follow from the proof of [21, Prop. 6.7]; the second inclusion gives $L_{m\lambda+\mu} = L_{m\lambda+\mu}(L_\lambda) = L_\mu(L_\lambda)$. Let T be a maximal torus of P_λ such that $\lambda, \mu \in Y(T)$. For sufficiently large m , we have $\langle m\lambda+\mu, \alpha \rangle > 0$ for all $\alpha \in \Psi(R_u(P_\lambda), T)$, whence $R_u(P_{m\lambda+\mu}) \supseteq R_u(P_\lambda)$. Since $P_\lambda = L_\lambda \ltimes R_u(P_\lambda)$, it follows that $P_{m\lambda+\mu} = (P_{m\lambda+\mu} \cap L_\lambda) \ltimes R_u(P_\lambda) = P_{m\lambda+\mu}(L_\lambda) \ltimes R_u(P_\lambda) = P_\mu(L_\lambda) \ltimes R_u(P_\lambda)$.

(ii). We can write $P = P_\lambda$ and $L = L_\lambda$ for some $\lambda \in Y(G)$. If $\mu \in Y(L)$, then $\lambda(k^*)$ and $\mu(k^*)$ commute, so $P_\mu(L) \ltimes R_u(P)$ is an R -parabolic subgroup of G by part (i). Conversely, let $\mu \in Y(G)$ such that $P_\mu \subseteq P$. Let T be a maximal torus of L . If P_μ is of the form $P' \ltimes R_u(P)$ for some R -parabolic subgroup P' of L , then any P -conjugate of P_μ is also of this form; thus, since any two maximal tori of P are P -conjugate, we can assume that $\mu(k^*) \subseteq T$. Since $P_\mu \subseteq P$, we have $P_\mu^0 \subseteq P^0$, whence $R_u(P) = R_u(P^0) \subseteq R_u(P_\mu^0) = R_u(P_\mu)$ (the middle inclusion is a standard result for connected groups). Thus $P_\mu = (P_\mu \cap L) \ltimes R_u(P) = P_\mu(L) \ltimes R_u(P)$.

(iii). Write $P = P_\lambda$, $L = L_\lambda$, $Q = P_\mu$, and $M = L_\mu$, where $\lambda, \mu \in Y(G)$. Since T is a maximal torus of L and $\lambda(k^*) \subseteq Z(L)^0$, we have $\lambda(k^*) \subseteq T$, whence $\lambda(k^*) \subseteq P \cap Q$. Since $P \cap Q$ is closed, it follows that $P \cap Q$ is c_λ -stable, so we have $P \cap Q = (L \cap Q)(R_u(P) \cap Q)$. By a similar argument, $L \cap Q = (L \cap M)(L \cap R_u(Q))$ and $R_u(P) \cap Q = (R_u(P) \cap M)(R_u(P) \cap R_u(Q))$, and the product decomposition (6.3) of $P \cap Q$ follows. It is easily checked that

$$V := (L \cap R_u(Q))(R_u(P) \cap M)(R_u(P) \cap R_u(Q))$$

is a normal subgroup of $P \cap Q$ (note that $[L \cap R_u(Q), R_u(P) \cap M] \subseteq R_u(P) \cap R_u(Q)$), and V , being constructible, is closed. Now $R_u(P) \cap R_u(Q)$ is unipotent and $V/(R_u(P) \cap R_u(Q)) \cong (L \cap R_u(Q)) \times (R_u(P) \cap M)$ is unipotent, so V is unipotent, and $V \subseteq G^0$, as $R_u(P)$ and $R_u(Q)$ are contained in G^0 . As V is normalized by T , it is connected thanks to [2, Prop. 14.4(2a)]. Thus $V \subseteq R_u(P \cap Q)$ and $R_u(P \cap Q) = ((L \cap M) \cap R_u(P \cap Q))V$. Now $L \cap M = L_\lambda(L_\mu)$ is an R -Levi subgroup of G , by part (i), so $(L \cap M)^0 = (L \cap M) \cap G^0$ is a connected reductive group and $(L \cap M) \cap R_u(P \cap Q) = \{1\}$. We deduce that $V = R_u(P \cap Q)$, as required. \square

The next result follows immediately from part (ii) of Lemma 6.2.

Corollary 6.4. *Let P be an R -parabolic subgroup of G with R -Levi subgroup L , and let H be a closed subgroup of L . Then P is minimal amongst the R -parabolic subgroups of G containing H if and only if H is L -irreducible.*

Corollary 6.5. *Let P be an R -parabolic subgroup of G and let T be a maximal torus of P . Then there exists $\lambda \in Y(T)$ such that $P = P_\lambda$. Moreover, L_λ is the unique R -Levi subgroup of P that contains T .*

Proof. Choose $\lambda \in Y(G)$ such that $P = P_\lambda$. Since maximal tori in P are P -conjugate, there exists $x \in P$ such that $x\lambda(k^*)x^{-1} \subseteq T$. We have $T \subseteq L_{x\lambda}$, and it is easily checked that

$P_{x\cdot\lambda} = xP_\lambda x^{-1} = P_\lambda = P$. If L is another R-Levi subgroup of P containing T , then it follows from Lemma 6.2(iii), setting $P = Q$ and $M = L_\lambda$, that $L = L_\lambda$. \square

Corollary 6.6. *Let P and Q be R-parabolic subgroups of G with $P \subseteq Q$ and let L be an R-Levi subgroup of P . Then there is a unique R-Levi subgroup M of Q such that $L \subseteq M$.*

Proof. Choose $\mu \in Y(G)$ such that $P = P_\mu$ and $L = L_\mu$. Choose a maximal torus T of Q with $\mu(k^*) \subseteq T$. By Corollary 6.5, we can find $\lambda \in Y(T)$ such that $Q = P_\lambda$. Applying Lemma 6.2(iii) to the R-parabolic subgroups P and Q and their respective R-Levi subgroups L_μ and L_λ , yields $L_\mu \subseteq L_\lambda$, so we can take $M = L_\lambda$. Uniqueness follows from Corollary 6.5, as any R-Levi subgroup contains a maximal torus of G . \square

Corollary 6.7. *Let P be an R-parabolic subgroup of G , let L be an R-Levi subgroup of P , and let M be a closed subgroup of P . Then M is an R-Levi subgroup of P if and only if M is $R_u(P)$ -conjugate to L .*

Proof. Write $P = P_\lambda$, $L = L_\lambda$ for some $\lambda \in Y(G)$. Clearly, if $u \in R_u(P)$, then $uL_\lambda u^{-1} = L_{u\cdot\lambda}$ and $P_{u\cdot\lambda} = uP_\lambda u^{-1} = P_\lambda$, so uLu^{-1} is an R-Levi subgroup of P . Conversely, suppose that M is an R-Levi subgroup of P , say $M = L_\mu$ with $P = P_\mu$ for some $\mu \in Y(G)$. Since maximal tori of P are P -conjugate and $P = R_u(P)M$, the R-Levi subgroups L and uMu^{-1} contain a common maximal torus for some $u \in R_u(P)$. Now Corollary 6.5 implies that $uMu^{-1} = L$. \square

Corollary 6.8. *Let P be an R-parabolic subgroup of G and let L be a Levi subgroup of P^0 . Then there exists $\lambda \in Y(G)$ such that $P = P_\lambda$ and $L = L_\lambda^0$.*

Proof. Let T be a maximal torus of L . By Corollary 6.5, there exists $\lambda \in Y(T)$ such that $P = P_\lambda$. As L and L_λ^0 are both Levi subgroups of P^0 containing the maximal torus T , Corollary 6.5 implies that $L = L_\lambda^0$. \square

The following result is the generalization of [4, Prop 4.4(b)] for non-connected G .

Corollary 6.9. *If P and Q are R-parabolic subgroups of G , then $(P \cap Q)R_u(Q)$ is an R-parabolic subgroup of G . Moreover, $(P \cap Q)R_u(Q) = Q$ if and only if P contains an R-Levi subgroup of Q .*

Proof. By a standard result for connected groups, [4, 2.4], $P \cap Q$ contains a maximal torus T of G . We can write $P = P_\mu$ and $Q = P_\lambda$ for some $\lambda, \mu \in Y(T)$, by Corollary 6.5. We have $(P \cap Q)R_u(Q) = (P_\mu \cap L_\lambda) \ltimes R_u(P_\lambda) = P_\mu(L_\lambda) \ltimes R_u(P_\lambda)$, and this is an R-parabolic subgroup of G by Lemma 6.2(ii). From the first equality we see that if $(P \cap Q)R_u(Q) = Q$, then $L_\lambda \subseteq P$. Conversely, it is clear that if P contains an R-Levi subgroup of Q , then $(P \cap Q)R_u(Q) = Q$. \square

Corollary 6.10. *If S is a torus of G , then $C_G(S)$ is an R-Levi subgroup of G .*

Proof. If S is central in G , then $C_G(S) = G = P_\lambda$, where λ is the trivial cocharacter of G . Otherwise we can find $\lambda \in Y(S)$ such that $\lambda(k^*) \not\subseteq Z(G)$. Then L_λ is a proper non-connected reductive subgroup of G containing S , and $C_G(S) = C_{L_\lambda}(S)$. By noetherian induction on closed subgroups of G , we can assume that $C_G(S)$ is an R-Levi subgroup of L_λ , say $C_G(S) = L_\mu(L_\lambda)$ for some $\mu \in Y(L_\lambda)$. We have $L_\mu(L_\lambda) = L_{m\lambda+\mu}$ for some $m \in \mathbb{N}$ by Lemma 6.2(i), as required. \square

Lemma 6.11. *Let P be an R -parabolic subgroup of G with R -Levi subgroup L . Then there exists a unique R -parabolic subgroup P^- of G such that $P \cap P^- = L$.*

Proof. Write $P = P_\lambda$, $L = L_\lambda$ for some $\lambda \in Y(G)$. It is simple to check that $P_\lambda \cap P_{-\lambda} = L_\lambda$. Uniqueness follows easily from uniqueness in the connected case. \square

Lemma 6.12. (i) *Let H be a closed subgroup of G^0 . Then H is G -completely reducible if and only if H is G^0 -completely reducible.*
(ii) *If H is a G -completely reducible subgroup of G , then $H \cap G^0$ is G^0 -completely reducible.*

Proof. (i). Suppose that H is G -cr. Let P be a parabolic subgroup of G^0 with $H \subseteq P$. Then H is contained in $N_G(P)$, which is an R -parabolic subgroup of G by Proposition 6.1. Since H is G -cr, there is an R -Levi subgroup L of $N_G(P)$ with $H \subseteq L$, so $H \subseteq L^0$, which is a Levi subgroup of $N_G(P)^0 = P$. Conversely, suppose that H is G^0 -cr. Let P be an R -parabolic subgroup of G with $H \subseteq P$. Then $H \subseteq P^0$, so there exists a Levi subgroup L of P^0 with $H \subseteq L$, as H is G^0 -cr. By Corollary 6.8, we have $L = L_\lambda^0$ for some $\lambda \in Y(G)$ with $P = P_\lambda$, so we are done.

(ii). If H is G -cr, then $H \cap G^0$ is G -cr, by the non-connected version of Theorem 3.10. Now apply part (i). \square

Example 6.13. In general, the converse of Lemma 6.12(ii) is false. For example, let $C_p = \langle a \mid a^p \rangle$ be the cyclic group of order $p = \text{char } k$, and let $G = C_p \times \text{SL}_2(k)$. Let $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and set $H = \langle a\gamma \rangle$. Then $H \cap G^0 = \{1\}$ is G^0 -cr and $H \subseteq P_\lambda$, where $\lambda \in Y(G)$ is given by $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, but H is not contained in any R -Levi subgroup of P_λ .

Lemma 6.14. *Let N be a closed normal subgroup of G . Then there exists a closed subgroup M of G such that $G = MN$, $M \cap N$ is finite, and M^0 commutes with N^0 .*

Proof. Clearly, we can assume that N is connected. Thus $[N, N]$ is the product of certain simple factors of G^0 . Let M_1 be the product of the remaining simple factors of G^0 . As G permutes the simple factors of G^0 and $[N, N]$ is normalized by G^0 , M_1 is also normalized by G ; moreover, N and M_1 commute. By the proof of [21, Prop. 3.2], there exists a finite subgroup F of G such that $G = FG^0$. The torus $S := (Z(G^0) \cap N)^0$ is normal in G , so Lemma 2.1 of [21] implies that there exists a subtorus S_1 of $Z(G^0)^0$ such that S_1 is normal in G , $SS_1 = Z(G^0)^0$, and $S \cap S_1$ is finite. Set $M = FM_1S_1$. It is straightforward to check that M has the required properties. \square

We now indicate which results from the earlier sections hold for non-connected groups. Subsections 6.2–6.5 below deal with the material in Sections 2–5 respectively. Our convention, except for in Subsection 6.4, is that all of the results and discussion go through to the non-connected case unless otherwise stated below, apart from obvious exceptions such as examples involving connected groups. If the proof of a result is significantly different in the non-connected case, then we describe the necessary modifications.

6.2. Geometric invariant theory and basic properties of G -cr subgroups. (See Section 2). Everything in Subsection 2.3 goes over to the non-connected case with minor modifications, cf. [21, §4], [22, §1]; for example, if G acts on an affine variety X and $x \in X$, then $G \cdot x$ is closed if and only if $G^0 \cdot x$ is closed, so the Hilbert–Mumford Theorem 2.3

generalizes immediately to non-connected G . A proof that a strongly reductive subgroup of G is non-connected reductive may be found in [21, §6]. It follows from this and the non-connected version of Theorem 3.1 that a G -cr subgroup of G is non-connected reductive. The non-connected version of Lemma 2.6 follows from the proof of [21, Prop. 6.6]. In Lemma 2.10, the finiteness of the number of conjugacy classes of R-parabolic subgroups and R-Levi subgroups follows from [21, Prop. 5.2(e)] and Corollary 6.7.

Lemma 2.11 holds in the non-connected case. To see this, note that for $f : G_1 \rightarrow G_2$ an isogeny of non-connected reductive groups, $\ker f$ is a finite normal subgroup of G_1 , so it is centralized by G_1^0 ; this implies that $\ker f$ is contained in any R-Levi subgroup of G_1 , so part (ii) follows from part (i) as in the connected case. The argument given shows that $f(L_\lambda) = L_\mu$ and that L_λ is proper if and only L_μ is proper. We have $f(P_\lambda^0) = P_\mu^0$ from the connected case, so $f(P_\lambda) = f(L_\lambda P_\lambda^0) = L_\mu P_\mu^0 = P_\mu$, and (i) holds. Part (iv) follows from the connected case, because $R_u(P_\lambda)$ and $R_u(P_\mu)$ are connected.

Lemma 2.12 is more complicated for non-connected G : the underlying problem is that a normal torus of G need not be central. Part (i) of Lemma 2.12 holds in the non-connected setting, but part (ii)(b) does not for the G_2 -ir and G_2 -ind cases. For let $G = C_2 \ltimes k^*$, where $C_2 = \langle a \mid a^2 \rangle$ and a acts on k^* by $a \cdot t = t^{-1}$, and let $f : G \rightarrow G/G^0$ be the canonical projection. Then clearly G^0 is contained in a proper R-Levi subgroup of G , but $f(G^0) = \{1\}$ is trivially G/G^0 -ir.

The rest of Lemma 2.12 holds in the non-connected case, but we need a different proof. Let $f : G_1 \rightarrow G_2$ be a surjective homomorphism of non-connected reductive groups, with kernel N . Let $M \subseteq G_1$ satisfy the conclusion of Lemma 6.14 with respect to N . Let $g : M \rightarrow G_2$ be the restriction of f ; note that g is an isogeny.

First we need a generalization of Lemma 2.11.

Lemma 6.15. *Let $\lambda \in Y(M)$ and set $\mu = f \circ \lambda$. Then*

- (i) $f(P_\lambda(G_1)) = P_\mu(G_2)$, $f(L_\lambda(G_1)) = L_\mu(G_2)$;
- (ii) $f^{-1}(P_\mu(G_2)) = P_\lambda(G_1)$, $f^{-1}(L_\mu(G_2)) = L_\lambda(G_1)$;
- (iii) $R_u(P_\lambda(G_1)) = R_u(P_\lambda(M))$;
- (iv) $f(R_u(P_\lambda(G_1))) = R_u(P_\mu(G_2))$.

Proof. By the non-connected version of Lemma 2.11, we can assume that N is connected. Then N commutes with M^0 , so N lies in $L_\lambda(G_1)$. To prove (ii), therefore, it suffices to prove (i). Clearly, $P_\lambda(G_1) = P_\lambda(M)N$, so $f(P_\lambda(G_1)) = f(P_\lambda(M)) = P_\mu(G_2)$, by the non-connected version of Lemma 2.11. A similar argument gives $f(L_\lambda(G_1)) = L_\mu(G_2)$, as required. If $u \in R_u(P_\lambda(G_1))$, then, writing $u = mn$ with $m \in M$, $n \in N$, we have $1 = c_\lambda(u) = c_\lambda(m)c_\lambda(n) = c_\lambda(m)n$, whence $n = c_\lambda(m)^{-1} \in M$. Thus $u \in R_u(P_\lambda(G_1)) \cap M = R_u(P_\lambda(M))$, and part (iii) is proved. Part (iv) now follows from (iii) and the non-connected analogue of Lemma 2.11(iv). \square

Now suppose that H_1 is G_1 -cr. Suppose that $f(H_1) \subseteq P_\mu(G_2)$, where $\mu \in Y(G_2)$. Write $\mu = f \circ \lambda$, where $\lambda \in Y(M)$. Then $f^{-1}(f(H_1)) \subseteq f^{-1}(P_\mu(G_2)) = P_\lambda(G_1)$, by Lemma 6.15(ii), so $H_1 \subseteq P_\lambda(G_1)$. Since H_1 is G_1 -cr, H_1 lies in an R-Levi subgroup of $P_\lambda(G_1)$, so by Corollary 6.7, we have $H_1 \subseteq L_{u \cdot \lambda}(G_1)$ for some $u \in R_u(P_\lambda(G_1))$. Now $u \in R_u(P_\lambda(M))$, by Lemma 6.15(iii), so $u \cdot \lambda \in Y(M)$. Thus $f(H_1) \subseteq f(L_{u \cdot \lambda}(M)) = L_{f(u) \cdot \mu}(G_2)$, by Lemma 6.15(i); moreover, $P_{f(u) \cdot \mu}(G_2) = f(P_{u \cdot \lambda}(G_1)) = f(P_\lambda(G_1)) = P_\mu(G_2)$ (Lemma 6.15(i)), so $L_{f(u) \cdot \mu}(G_2)$ is an R-Levi subgroup of $P_\mu(G_2)$. It follows that $f(H_1)$ is G_2 -cr.

Suppose that H_1 is G_1 -ind. If $f(H_1) \subseteq L_\mu(G_2)$ for some $\mu \in Y(G_2)$ with $L_\mu(G_2) \neq G_2$ then, picking $\lambda \in Y(M)$ such that $\mu = f \circ \lambda$, we have $H_1 \subseteq f^{-1}(f(H_1)) \subseteq f^{-1}(L_\mu(G_2)) = L_\lambda(G_1)$, by Lemma 6.15(ii). By Lemma 2.11(iii), $L_\lambda(M) \neq M$, so $\lambda(k^*)$ does not centralize M , so $\lambda(k^*)$ does not centralize G_1 , so $L_\lambda(G_1) \neq G_1$. But this contradicts the G_1 -indecomposability of H_1 . We deduce that $f(H_1)$ is G_2 -ind.

It now follows that if H_1 is G_1 -ir then $f(H_1)$ is G_2 -ir. This completes the proof of Lemma 2.12(ii)(a) in the non-connected case.

Now we show that if f is non-degenerate and H_1 is a closed subgroup of G_1 such that $f(H_1)$ is G_2 -cr, then H_1 is G_1 -cr. By Lemma 2.11(i), we can assume that N is a torus. Thus N lies in every R-Levi subgroup of G_1 , so without loss we can assume that $H_1 \subseteq M$. Let $\lambda \in Y(G_1)$ with $H_1 \subseteq P_\lambda(G_1)$. Then $f(H_1) \subseteq P_\mu(G_2)$, where $\mu := f \circ \lambda$. As $f(H_1)$ is G_2 -cr, there is an R-Levi subgroup L of $P_\mu(G_2)$ with $f(H_1) \subseteq L$.

Choose $\sigma \in Y(M)$ such that $g \circ \sigma = \mu$. We have $\lambda(k^*) \subseteq \sigma(k^*)N$; since $N \subseteq Z(G_1^0)$, there exists $\tau \in Y(N)$ such that $\lambda = \sigma + \tau$. By Lemma 6.15(iv) and Corollary 6.7, there exists $u \in R_u(P_\sigma(M)) = R_u(P_\lambda(M)) \subseteq R_u(P_\lambda(G_1))$ such that $f(H_1) \subseteq L_{f(u) \cdot \mu}(G_2)$. Thus, replacing λ by $u \cdot \lambda$ if necessary, we can assume that $f(H_1) \subseteq L_\mu(G_2)$. We have $H_1 \subseteq g^{-1}(L_\mu(G_2)) = L_\sigma(M)$, by Lemma 2.11(ii). We have $\lambda \in Y(L_\sigma(G_1))$ and

$$H_1 \subseteq P_\lambda(G_1) \cap L_\sigma(M) \subseteq P_\lambda(G_1) \cap L_\sigma(G_1) = P_\lambda(L_\sigma(G_1)) = P_\tau(L_\sigma(G_1)).$$

Now $L_\sigma(G_1)^0 \subseteq P_\tau(L_\sigma(G_1))$, thus $R_u(P_\tau(L_\sigma(G_1))) = \{1\}$, and so $P_\tau(L_\sigma(G_1)) = L_\tau(L_\sigma(G_1))$. Thus $H_1 \subseteq L_\tau(L_\sigma(G_1)) \subseteq L_\lambda(G_1)$. We conclude that H_1 is G_1 -cr. This completes the proof of the remaining parts of Lemma 2.12 in the non-connected case.

For Propositions 2.13 and 2.16, see [21, Prop. 8.3]. Proposition 2.15 for non-connected G follows from the connected case, Lemma 6.12 and the non-connected version of Theorem 3.1.

6.3. G -complete reducibility. (See Section 3). The proofs of Theorem 3.1 and Corollary 3.5 go through from the connected case to the non-connected case, replacing various results concerning parabolic subgroups with their counterparts from Subsection 6.1 for R-parabolic subgroups. For example, we use Lemma 6.2(ii) instead of [4, Prop. 4.4(c)], Corollary 6.9 instead of [4, Prop. 4.4(b)] and (6.3) instead of (3.2).

Proposition 3.19 generalizes (with the same proof) to give the result for non-connected G and H . For the proof of Proposition 3.12, note that if M is a closed subgroup of G , then G/M is affine if and only if $G^0/(G^0 \cap M)$ is affine (cf. [31, §1]). By virtue of these extensions, we now obtain Theorem 3.14 immediately as a special case of Proposition 3.19 — the separate proof in Section 3 is not needed.

The first assertion of Corollary 3.18 fails in the non-connected case, by the counterexamples below to Lemma 3.38. The second assertion, however, still holds. For suppose that H is not G -cr. By Remark 2.9 and Corollary 3.7, we can assume that H is topologically finitely generated, say by h_1, \dots, h_n , and that the orbit $G \cdot (h_1, \dots, h_n)$ is not closed in G^n . By the Hilbert–Mumford–Kempf Theorem (see [14, Cor. 3.5]), there exists $\lambda \in Y(G)$ such that $H \subseteq P_\lambda$, $C_G(H) \subseteq N_G(P_\lambda)$, $G \cdot (c_\lambda(h_1), \dots, c_\lambda(h_n))$ is closed but $G \cdot (h_1, \dots, h_n)$ is not. This implies that $P_\lambda \neq L_\lambda$, whence $R_u(P_\lambda) \neq \{1\}$, whence $P_\lambda^0 \neq G^0$, whence $N_G(P_\lambda^0)$ is a proper R-parabolic subgroup of G , by Proposition 6.1. But this is impossible, since $C_G(H)$ is G -ir.

An immediate corollary of Proposition 3.20 is the following: if H is a subgroup of G containing G^0 , then H is G -cr.

Corollary 3.22 holds in the non-connected case, although the proof given in Section 3 does not work, because $C_G(S)$ need not be connected. Here is an alternative proof. Let L be an R-Levi subgroup of G . Recall that $L = C_G(Z(L)^0)$. Let K be a closed subgroup of L and let S be a maximal torus of $C_G(K)$ with $Z(L)^0 \subseteq S$. Then $S \subseteq L$, so S is a maximal torus of $C_L(K)$ and $C_G(S) = C_L(S)$. The desired result now follows from the non-connected analogue of Corollary 3.5.

We do not know whether Theorem 3.26 and Remark 3.30 hold in the non-connected case; the proof of [20, Thm. 4.5] does not generalize.

We extend the definition of a reductive pair to non-connected G and H in the obvious way. Example 3.33 is valid in the non-connected case: for H is generated by H^0 and $N_H(T)$, and $N_H(T)$ permutes $\Psi(G, T) \setminus \Psi(H, T)$. Also Example 3.37 is valid for non-connected H with H^0 a simple group of adjoint type.

Lemma 3.38 fails in the non-connected case: e.g., if G is an abelian p -group, then $Z(G)$ itself is not linearly reductive. It is not even true in general that $C_G(H)/Z(G)$ is linearly reductive: take G to be $C_2 \ltimes (M \times M)$, where $p = 2$, M is a connected simple group and C_2 acts by permuting the factors, and take H to be the copy of M diagonally embedded in G^0 .

We consider the material in Subsection 3.7 for connected G only. Theorem 3.46, however, holds in the non-connected case.

6.4. Complete reducibility in G and the building of G^0 . (See Section 4). We will not construct a simplicial complex using the R-parabolic subgroups of G for non-connected G . Instead we regard G as a subgroup of $\text{Aut } X$ ($X := X(G^0)$): we consider the action of G on the spherical building X induced by the conjugation action of G on G^0 . For a subgroup H of G , it need no longer be the case that X^H is a subcomplex of X . However, we can view the subspace X^H as a subcomplex of the barycentric subdivision of X , see [38, 2.3.1].

It is still possible to ask whether the subspace X^H is X -completely reducible or not. The following proposition allows us to extend the building-theoretic interpretation of G -complete reducibility to this situation.

Proposition 6.16. *Let H be a subgroup of G . Then H is G -completely reducible if and only if every simplex in X^H has an opposite in X^H .*

Proof. Recall that for any parabolic subgroup Q of G^0 , $N_G(Q)^0 = N_{G^0}(Q) = Q$, see the note following Proposition 6.1. Now suppose that H is G -cr. Let Q be a parabolic subgroup of G^0 fixed by H . Then $H \subseteq N_G(Q)$, which is an R-parabolic subgroup of G , by Proposition 6.1. Since H is G -cr, there exists $\lambda \in Y(G)$ with $N_G(Q) = P_\lambda$ and $H \subseteq L_\lambda = P_\lambda \cap P_{-\lambda}$. But then H fixes $P_{-\lambda}^0$, which is a parabolic subgroup of G^0 opposite to $P_\lambda^0 = Q$. Thus every simplex in X^H has an opposite in X^H .

Conversely, suppose that every simplex in X^H has an opposite in X^H . Let P be an R-parabolic subgroup of G containing H . Then $Q = P^0$ is an H -fixed simplex of X , so there exists an H -fixed simplex Q^- of X opposite Q . Let $M = Q \cap Q^-$ be the common Levi subgroup of Q and Q^- . Set $P' = N_G(Q^-)$. By Corollary 6.8, there exist R-Levi subgroups L and L' of P and P' respectively such that $L \cap G^0 = L' \cap G^0 = M$. Then

$$P \cap P' = (L \cap L')(L \cap R_u(P'))(L' \cap R_u(P))(R_u(P) \cap R_u(P')),$$

by Lemma 6.2(iii). However, $L \cap R_u(P') = M \cap R_u(Q^-) = \{1\}$, $L' \cap R_u(P) = M \cap R_u(Q) = \{1\}$, and $R_u(P) \cap R_u(P') = R_u(Q) \cap R_u(Q^-) = \{1\}$. Thus $H \subseteq P \cap P' = L \cap L' \subseteq L$, so H is G -cr, as claimed. \square

We now see that part of Theorem 4.1 holds for non-connected G : a subgroup H of G is G -cr if and only if the fixed point subspace X^H is X -cr. We are grateful to J.-P. Serre for drawing our attention to this extension.

Lemma 4.2 holds for non-connected G and also holds with $C_G(H)^0$ replaced everywhere by $C_G(H)$. Moreover, by the arguments in the proof of Proposition 4.3, the following is also true: if H is any closed (not necessarily connected) G -cr subgroup of G , then R-parabolic subgroups of $HC_G(H)^0$ containing H correspond bijectively to R-parabolic subgroups of $C_G(H)^0$, and the same is true replacing $C_G(H)^0$ everywhere with $C_G(H)$.

When the action of H is completely reducible and type-preserving, then X^H is a subcomplex of X and the results due to Mühlherr also hold: i.e., X^H is a building if and only if it is thick, and the thickening of X^H is isomorphic to $X(C_{G^0}(H)^0) = X(C_G(H)^0)$.

6.5. Rationality questions. (See Section 5). Theorem 5.8 holds in the non-connected case, by [2, Prop. 1.2]: if M is a linear algebraic group defined over k' , then M^0 is also defined over k' . The existence of a $\text{Gal}(\bar{k}/k)$ -invariant length function on $Y(G)$ follows from the arguments of [14, §4] and [22, §1]. We use Corollary 6.5 instead of [41, Cor. 8.4.4]. Note that [14, Thm. 4.2] holds for non-connected G — cf. the discussion of the Hilbert–Mumford Theorem in Subsection 6.2 — and the argument of [15, Prop. 2.2] also works in the non-connected case.

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